INTRODUCTION TO BERKOVICH ANALYTIC SPACES

MICHAEL TEMKIN

1. Introduction

This paper is a preliminary version of the lecture notes that accompanies my introductory course on Berkovich analytic spaces at Summer School "Berkovich spaces" at Institut de Mathmatiques de Jussieu, 29 June – 9 July, 2010.

1.1. **Berkovich spaces and some history.**

1.1.1. *Naive non-archimedean analytic spaces.* Since non-archimedean complete real valued fields (e.g. $\mathbf{Q}_p$) were discovered in the beginning of the century, it was very natural to try to develop a theory of analytic spaces over such a field k analogously to the theory of complex analytic spaces. At least, one can naturally define analytic functions on a subset V of the naive affine line $A^1_k = k$ as the convergent power series on V. This allowed to introduce some naive k-analytic spaces, but the theory was not reach enough. Actually, a global theory of such varieties does not make too much sense because the topology of k is totaly disconnected. In particular, locally analytic (and even constant) functions do not have to be globally analytic.

1.1.2. *Rigid geometry.* In his study of elliptic curves with bad reduction over k, Tate discovered in 1960ies that these curves admit a natural uniformization by G_m. The latter was given as an abstract isomorphism of groups $k^\times/q^{\mathbf{Z}} \widetilde{\to} E(k)$, and even such expert as Grothendieck doubted at first that this is not an accidental brute force isomorphism. Tate suspected, however, that his isomorphism can be interpreted as an analytic one, and he had to develop a good global theory of non-archimedean analytic spaces to make this rigorous. This research resulted in Tate's definition of rigid geometry, whose starting idea was to simply forbid all bad open coverings (responsible for disconnectedness) and to shrink the set of analytic functions accordingly. As a result, one obtains a good theory of sheaves of analytic functions, but the underlying topological spaces have to be replaced with certain topologized (or Grothendieck) categories, also called G-topological spaces.

1.1.3. *Berkovich spaces.* More recently, few other approaches to "right" non-archimedean geometry were discovered: Raynaud's theory of formal models, Berkovich' analytic geometry and Huber's adic geometry. They all allow to define (nearly) the same categories of k-analytic spaces, but provide the analogs of rigid spaces with additional structures invisible in the classical Tate's theory. Also, they extend the category of rigid spaces in different directions. Here we only discuss Berkovich' theory, which was developed in [Ber1] and [Ber2]. In this theory classical rigid spaces are saturated with many new points (analogs of non-closed points of algebraic varieties), and the obtained spaces are honest topological spaces. In addition, the obtained topological spaces are rather nice (locally pathwise connected, for example).

Now, let us list few interesting features of Berkovich theory that distinguish it from all other approaches. First, one can work with all positive real numbers almost as well as with the values of $|k^\times|$. In particular, one can study rings of power series with radii of convergence linearly independent of $|k^\times|$. The latter fact allows to include the case of a trivially valued k into the theory, and the theory of such k-analytic fields was already applied to classical problems of algebraic geometry. Another interesting feature is that in a similar fashion one can develop (an equivalent form of) the usual theory of complex analytic spaces. Moreover, one can define Berkovich spaces that include both archimedean and non-archimedean worlds, for example, the affine line over $(\mathbf{Z}, |\ |_\infty)$.

1.2. **Structure of the notes.** We do not aim to prove all results discussed in these notes (and this is impossible in a six lecture long course). Our goal is to make the reader familiar with basic definitions, constructions, techniques and results of non-archimedean analytic geometry. Therefore, we prefer to formulate difficult results as Facts, and in some cases we discuss main ideas of their proofs in Remarks. Easy corollaries from these results (that may themselves be important pieces of the theory) are then suggested as exercises to the reader. Many of exercises are provided with hints, but it may be worth to first try to solve them independently (especially, those not marked by an asterisk).

1.2.1. *Overview.* The course is divided to five sections as follows. First, we study in §2 semi-norms, norms and valuations, basic operations with these objects, Banach rings and their spectra. Then we describe the structure of $\mathcal{M}(\mathbf{Z})$, and after that we switch completely to the non-archimedean world. We finish the section with describing affine line over an algebraically closed non-archimedean field. In §3 we introduce k-affinoid algebras and spaces and study their basic properties. In the next section, this local theory is used to introduce and study global k-analytic spaces. Relations of k-analytic spaces with other categories are studied in §4. This includes, analytification of algebraic varieties and GAGA, generic fibers of formal k°-schemes and Raynaud's theory, and some discussion of rigid and adic geometries. Finally, in §5 we study k-analytic curves in details. In particular, we describe their local and global structure and explain how this is related to the stable reduction theorem for formal k°-curves.

1.2.2. *References and other sources.* The main references that helped me to prepare the course are [Ber1], [Ber2] and [Ber3]. The first two are a book and a large article in which the non-archimedean analytic spaces were introduced. The third one is a lecture notes of an analogous introductory course given by Berkovich in Trieste in 2009. I recommend the third source as an alternative (and shorter) expositional introductory text. It is worth to note that the first three sections of [Ber3] and of these notes are parallel, but the exposition is often but not always rather different. Also, this text contains much more exercises and remarks, and this seriously increases its length. Finally, the reader may wish to consult lecture notes [Con] on non-archimedean geometry (including the rigid geometry) by Brian Conrad.

1.2.3. *Conventiones.* Throughout these notes *ring* always means a commutative ring with unity. For any field k by k^s and k^a we denote its separable and algebraic closures, respectively. We will use underline to denote finite tuples of real numbers or of coordinates. For example, a polynomial ring $k[T_1, \dots, T_n]$ will often be denoted as $k[\underline{T}]$ where $\underline{T}$ is the tuple $(T_1, \dots, T_n)$ of coordinates. Also we will use the notation $\underline{T}^i = T_1^{i_1} \dots T_n^{i_n}$ for $i \in \mathbf{N}^n$. For example, a power series $f(\underline{T}) \in k[[\underline{T}]]$ can be uniquely written as $\sum_{i \in \mathbf{N}^n} a_i \underline{T}^i$ with $a_i \in k$.

Contents

2. Norms, valuations and Banach rings

2.1. **Seminorms.**

2.1.1. *Seminormed groups.*

Definition/Exercise 2.1.1. (i) A *seminorm* on an abelian group A is a function $|\ | : A \to \mathbf{R}_+$ which is *sub-additive*, i.e. $|a+b| \le |a| + |b|$, and satisfies $|0| = 0$ and $|-a| = |a|$. A seminorm is a *norm* if its kernel is trivial. If the seminorm is fixed then we call A a *seminormed group*.

(ii) The morphisms in the category of seminormed abelian groups are *bounded homomorphism*, i.e. homomorphisms $\phi : A \to B$ such that $\|\phi(a)\| \le C|a|$ for some fixed constant $C = C(\phi)$. In particular, $(A, |\ |)$ and $(A, \|\ \|)$ are isomorphic if and only if the seminorms $|\ |$ and $\|\ \|$ are *equivalent*, i.e. there exists a constant $C > 0$ such that $|a| \le C\|a\|$ and $\|a\| \le C|a|$ for any $a \in A$.

(iii) Any quotient A/H possesses a *residue seminorm* $\|\ \|$ given by $\|a + H\| = \inf_{h\in H} |a + h|$. A homomorphism of seminormed groups $\phi : A \to B$ is *admissible* if the residue seminorm on $\phi(A)$ is equivalent to the seminorm induced from B.

(iv) We provide a seminormed ring A with the *semimetric* $d(a, b) = |a - b|$. The induced *seminorm topology* is the weakest topology for which the balls $B_{a,r} = \{x \in A|\ |x-a| < r\}$ are open. This topology distinguishes points (i.e. is T_0) if and only if the seminorm is a norm. Two seminorms are equivalent if and only if their induced topologies coincide. Any bounded homomorphism is continuous with respect to the seminorm topologies (see also Exercise 2.2.3).

(v) The *separated completion* $\widehat{A}$ of a seminormed group A is the set of equivalence classes of Cauchy sequences in A. Use continuity to extend the group structure to $\widehat{A}$ and show that $\widehat{A}$ is a normed group, the natural map $A \to \widehat{A}$ is an admissible homomorphism (called the *separated completion homomorphism*) and its kernel is $\mathrm{Ker}(|\ |)$. In particular, $A/\mathrm{Ker}(|\ |)$ is a normed group with respect to the residue seminorm.

Remark 2.1.2. Usually, we will simply say "completion" in the sequel. Sometimes we will say "separated completion" when it is important to remember that the completion homomorphism may have a kernel.

2.1.2. *Seminormed rings and modules.*

Definition/Exercise 2.1.3. (i) A *seminorm* (reps. *norm*) on a ring A is a seminorm (resp. norm) on the additive group of A which is *submultiplicative*, i.e. $|ab| \le |a||b|$. If $|\ |$ is multiplicative, i.e. $|ab| = |a||b|$ and $|1| = 1$, then it is called a *real semivaluation* (resp. *real valuation*). If such a structure is fixed then the ring is called *seminormed*, *normed*, *real valued* or *real semivalued*, accordingly.

(ii) A seminorm on an A-module M is an additive seminorm $\|\ \|$ such that $\|am\| \le C|a|\|m\|$ for a fixed $C = C(M)$ and any $a \in A$ and $m \in M$.

(iii) Formulate and prove the analogs of all results/definitions from 2.1.1, including separated completions and admissible homomorphisms.

Remark 2.1.4. (i) General semivaluations on rings are defined similarly but with values in $\{0\} \sqcup \Gamma$, where Γ is a totally ordered abelian group.

(ii) When studying general semivaluations one usually does not distinguish between the *equivalent* ones, i.e. semivaluations with an ordered isomorphism $i : |A| \widetilde{\to} \|A\|$ such that $i \circ |\ | = \|\ \|$. This is the only reasonable possibility in the case that the group of values Γ is not fixed. On the other side, it is very important that we do distinguish equivalent but not equal real semivaluations.

(iii) The valuation terminology is not unified in the literature. For example, in adic geometry of R.Huber, any semivaluation is called a valuation.

In the following exercise we provide some definitions, examples and constructions related to seminormed rings.

Definition/Example/Exercise 2.1.5. Let $(A, |\ |)$ be a normed ring (the constructions make sense for seminormed rings but we will not need that).

(i) The *spectral seminorm* $\rho = \rho_{\mathcal{A}}$ is the maximal power-multiplicative seminorm dominated by $|\ |$. Show that $\rho(f) = \lim_{n\to\infty} |f^n|^{1/n}$.

(ii) For a tuple of positive numbers $\underline{r} = (r_1, \dots, r_n)$ provide $A[T_1, \dots, T_n]$ with the norm

$$\|\sum_{i\in\mathbf{N}^n} a_i \underline{T}^i\|_{\underline{r}} = \sum_{i\in\mathbf{N}^n} |a_i|\underline{r}^i$$

and let $A\{\underline{r}^{-1}\underline{T}\} = A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$ denote its completion. This ring can be viewed as the ring of convergent power series over A with polyradius of convergence $\underline{r}$ (work this out: show that $A\{\underline{r}^{-1}\underline{T}\}$ is a subring of $\widehat{A}[[\underline{T}]]$ defined by natural convergence conditions).

(iii) If $(M, |\ |_M)$ and $(N, |\ |_N)$ are normed A-modules (resp. rings) then we provide $M \otimes_A N$ with the *tensor product seminorm* $\|x\| = \inf(\sum_{i=1}^n |m_i|_M |n_i|_N)$ where the infimum is taken over all representations of x of the form $x = \sum_{i=1}^n m_i \otimes n_i$. The separated completion of this seminormed module is denoted $M\widehat{\otimes}_A N$ and called the *completed tensor product* of modules (resp. rings). We will later see that the tensor product seminorm is often not a norm.

(iv) The *trivial semi-norm* $|\ |_0$ on a ring A sends $A \setminus \{0\}$ to 1. It is a valuation (resp. power-multiplicative) if and only if A is integral (resp. reduced).

(v) For any natural $n > 1$ define n-adic norm on $\mathbf{Q}$ so that $|x|_n = n^d$ where $d \in \mathbf{Z}$ is the minimal number with $xn^d \in \mathbf{Z}$. This norm is a valuation only when n is prime. The equivalence class of $|\ |_n$ depends only on the set $p_1, \dots, p_m$ of prime divisors of n. The completion $\mathbf{Q}_n$ with respect to $|\ |_n$ is called the ring of n-adic numbers. Show that $\mathbf{Q}_n = \oplus_{i=1}^m \mathbf{Q}_{p_i}$. In particular, completion does not preserve the property of being an integral domain. Show that $\mathbf{Q}_p$ is a field. It is called the *field of p-adic numbers.*

(vi) Define t-adic valuations on $k[t]$ analogously to the p-adic valuation (they are trivial on k and are uniquely determined by $r = |t| \in (0, 1)$). Show that $k[[t]]$ is the completion.

(vii) Ostrowski's theorem provides a complete list of real semivaluations on $\mathbf{Z}$: the trivial valuation $|\ |_0$, the p-adic valuations $|\ |_{p,r} = (|\ |_p)^r$ for any $r \in (0, \infty)$, the archimedean valuations $|\ |_{\infty,r} = (|\ |_\infty)^r$ for $r \in (0, 1]$ (where $|x|_\infty$ is the usual absolute value of x), and the semivaluations $|\ |_{p,\infty}$ that take $p\mathbf{Z}$ to 0 and everything else to 1.

Remark 2.1.6. There is a certain analogy, that will be used later, between ideals on rings and bounded semi-norms on semi-normed rings. Exercises (iv) and (v) above indicate that multiplicative (resp. power-multiplicative) semi-norms correspond to prime (resp. reduced) ideals. In the style of the same analogy, passing from a semi-normed ring $(A, |\ |)$ to $(A/\mathrm{Ker}(\rho_A), \rho_A)$ can be viewed as an analog of reducing the ring (i.e. factoring a ring by its radical). The elements in the kernel of ρ_A are called *quasi-nilpotent* elements.

2.2. Banach rings and their spectra.

2.2.1. *Banach rings, algebras and modules.*

Definition 2.2.1. (i) A *Banach ring* is a complete normed ring $\mathcal{A}$ (i.e. the completion homomorphism $\mathcal{A} \to \widehat{\mathcal{A}}$ is an isomorphism). A *Banach $\mathcal{A}$-algebra* is a Banach algebra $\mathcal{B}$ with a bounded homomorphism $\mathcal{A} \to \mathcal{B}$.

(ii) A *Banach $\mathcal{A}$-module* is a complete normed $\mathcal{A}$-module.

Instead of the polynomial rings and tensor products of modules, when working with Banach rings and modules we will use the convergent power series rings and completed tensor products.

Fact 2.2.2. The valuation of any complete real valued field k uniquely extends to any algebraic extension of k.

Example/Exercise 2.2.3. Let k be a complete real valued field and let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism of Banach k-modules (also called Banach k-spaces).

(i) Assume that k is not trivially valued (e.g. $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Q}_p$ or $\mathbf{C}((t))$). It follows from Banach open map theorem that f is bounded if and only if it is continuous.

(ii) Assume that k is trivially valued. Show that as an abstract ring, $k\{r^{-1}T\}$ is isomorphic to $k[T]$ when $r \geq 1$ and is isomorphic to $k[[T]]$ when $r < 1$. Conclude that Banach open map theorem fails for this k and give an example of continuous but not bounded homomorphism of k-algebras.

2.2.2. *The spectrum.* The analogy from Remark 2.1.6 suggests the following definition.

Definition 2.2.4. (i) The *spectrum* of a Banach ring $\mathcal{A}$ is the set $\mathcal{M}(\mathcal{A})$ of all bounded real semivaluations $|\ |_x$ on $\mathcal{A}$ (i.e. $|\ |_x \leq C|\ |$ for some C) provided with the weakest topology making continuous the maps $|f| : \mathcal{M}(\mathcal{A}) \to \mathbf{R}_+$ for all $f \in \mathcal{A}$. The latter maps take $|\ |_x$ to $|f|_x$ and usually we will use the notation $x \in \mathcal{M}(\mathcal{A})$ and $|f(x)|$ instead of $|\ |_x$ and $|f|_x$.

(ii) For any point $x \in \mathcal{M}(\mathcal{A})$ the kernel of $|\ |_x$ is a prime ideal and hence $A/\mathrm{Ker}(|\ |_x)$ is an integral valued ring. The completed fraction field of this ring is called the *completed residue field of* x and we denote it as $\mathcal{H}(x)$. The bounded character corresponding to x will be denoted $\chi_x : \mathcal{A} \to \mathcal{H}(x)$.

The following exercise shows that the definition of $\mathcal{M}$ is analogous to the definition of Spec.

Exercise 2.2.5. The points of $\mathcal{M}(\mathcal{A})$ are the isomorphism classes of bounded homomorphisms $\chi : \mathcal{A} \to k$ whose image is a complete real valued field generated by the image of χ (i.e. $\mathrm{Im}(\chi)$ is not contained in a complete subfield of k).

Here are the basic facts about the spectrum. As one might expect, the general line of the proof is to construct enough points by use of Zorn's lemma.

Fact 2.2.6. (i) Let $\mathcal{A}$ be a Banach ring. The spectrum $X = \mathcal{M}(\mathcal{A})$ is compact, and it is empty if and only if $\mathcal{A} = 0$.

(ii) The maximum modulus principle: $\rho(f) = \max_{x \in X} |f(x)|$.

Exercise 2.2.7. (i) Extend $\mathcal{M}$ to a functor to topological spaces, that is, for any bounded homomorphism of Banach rings $\phi : \mathcal{A} \to \mathcal{B}$ construct a natural continuous map $\mathcal{M}(\phi) : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{A})$.

(ii) An element $f \in \mathcal{A}$ is invertible if and only if $\inf_{x \in X} |f(x)| > 0$.

(iii) If $\mathcal{A}$ is a Banach k-ring for a complete real valued field k then

$$\mathcal{M}(\mathcal{A}\widehat{\otimes}_k k^a)/\mathrm{Gal}(k^s/k) \widetilde{\to} \mathcal{M}(A)$$

where k^a and k^s are provided with the extended valuation.

(iv)* Let $\mathcal{A}$ be finite over $\mathbf{Z}$. Show that up to equivalence there exists unique structure of a Banach $(\mathbf{Z}, |\ |_\infty)$-algebra on $\mathcal{A}$ and describe $\mathcal{M}(\mathcal{A})$. Analyse similarly the spectra of finite Banach $(k[T], |\ |_0)$-algebras there k is a field and $|\ |_0$ is the trivial valuation. (Hint: take all the scheme Spec($\mathcal{A}$). Keep all its closed points, and for any generic point x of a curve component replace x with all valuations on $k(x)$.)

2.2.3. *The affine space.*

Definition 2.2.8. The *n-dimensional affine space* over a Banach ring $\mathcal{A}$ is the set $\mathbf{A}^n_{\mathcal{A}} = \mathcal{M}(\mathcal{A}[T_1, \dots, T_n])$ of all real semivaluations on $\mathcal{A}[\underline{T}]$ that are bounded on $\mathcal{A}$ (i.e. the restriction of $| \ |_x$ to $\mathcal{A}$ is bounded). Naturally, $\mathbf{A}^n_{\mathcal{A}}$ is provided with the weakest topology making continuous each map $x \mapsto |f(x)|$ with $f \in \mathcal{A}$.

Exercise 2.2.9. The affine space $\mathbf{A}^n_{\mathcal{A}}$ is the union of closed $\mathcal{A}$-*polydiscs* $\mathcal{M}(\mathcal{A}\{\underline{r}^{-1}\underline{T}\})$ of *polyradius* $\underline{r} = (r_1, \dots, r_n)$. In particular, it is locally compact.

Remark 2.2.10. For a complete real valued field k one can provide $\mathbf{A}^n_k$ with the sheaf of analytic functions which are local limits of rational functions from $k(\underline{T})$. If $U \subset \mathbf{A}^n_k$ is open and V is a Zariski closed subset of U given by vanishing of analytic functions $f_1, \dots, f_n$, then factoring by the ideal generated by f_i one obtains a sheaf of analytic functions on V. Gluing such local models V with the sheaves of analytic functions one can construct a theory of k-analytic spaces without boundary. The advantage of this approach is that it works equally well over $\mathbf{C}$ and over $\mathbf{Q}_p$. The main disadvantage of this approach is that it does not treat well enough the cases of a trivially valued k and of analytic $\mathbf{Q}_p$-spaces with boundary. More details about the outlined approach can be found in [Ber1, §1.5] and [Ber3, §1.3]. We will use another approach to construct non-archimedean analytic spaces.

2.3. **Non-archimedean setting.**

2.3.1. *Strong triangle inequality.*

Definition 2.3.1. (i) A non-archimedean seminorm (resp. norm, semivaluation, etc.) is a seminorm $| \ |$ that satisfies the *strong triangle inequality* $|a + b| \le \max(|a|, |b|)$.

(ii) A *non-archimedean field* is a complete real valued field k whose valuation is non-arhimedean.

Example 2.3.2. (i) By Gel'fand-Naimark theorem any complete real valued field, excluding $\mathbf{R}$ and $\mathbf{C}$, is non-archimedean.

(ii) For any ring A its trivial seminorm is non-archimedean.

Exercise 2.3.3. Show that for an archimedean k one has that $\mathbf{A}^1_k \widetilde{\to} k$ where the image is provided with the valuation topology. (Hint: use Gel'fand-Naimark.)

In the sequel we will work only with non-archimedean seminorms, semivaluations, etc., so the word "non-archimedean" will usually be omitted. Let $\mathcal{A}$ be a non-archimedean Banach ring. The basic definitions should now be adjusted as follows.

Definition/Example/Exercise 2.3.4. (i) Check that $\mathcal{M}(\mathcal{A})$ is the set of all non-archimedean bounded semivaluations on $\mathcal{A}$.

(ii) The spectral seminorm $\rho_{\mathcal{A}}$ is non-archimedean.

(iii) In the definitions of $\mathcal{A}\{\underline{r}^{-1}\underline{T}\}$, $M\widehat{\otimes}_{\mathcal{A}}N$ and their norms we replace the sums with maxima. For example, $\|\sum_{i\in\mathbf{N}^n} a_i\underline{T}^i\|_{\underline{r}} = \max_{i\in\mathbf{N}^n} |a_i|\underline{r}^i$. Check that this (modified) norm is a valuation.

(v) Calculus student's dream: a sequence a_n in $\mathcal{A}$ is Cauchy if and only if $\lim_{n\to\infty} |a_n - a_{n+1}| = 0$. In particular, a series $\sum_{n=0}^{\infty} a_n$ converges in $\mathcal{A}$ if and only if $\lim_{n\to\infty} |a_n| = 0$.

At this point we fix a non-archimedean ground field k and start to develop non-archimedean analytic geometry over k. When developing this theory we will compare it from time to time to the classical theory of algebraic varieties over a field. Analogously to the latter theory, we will first introduce k-Banach algebras of topologically finite type and their spectra, called k-affinoid algebras and spaces. Then we will construct general k-analytic spaces by pasting k-affinoid ones. Despite this general similarity, many details in our theory are much subtler. We will try to indicate critical moments where the theories differ.

2.3.2. *Reduction ring.*

Definition 2.3.5. It follows from Exercise 2.3.4 that any non-archimedean ring $\mathcal{A}$ contains an open subring $\mathcal{A}^\circ = \{a \in \mathcal{A}|\ \rho(a) \le 1\}$ with an ideal $\mathcal{A}^{\circ\circ} := \{a \in \mathcal{A}|\ \rho(a) < 1\}$. The ring $\widetilde{\mathcal{A}} = \mathcal{A}^\circ/\mathcal{A}^{\circ\circ}$ is called the *reduction ring* of $\mathcal{A}$.

Example/Exercise 2.3.6. If k is a non-archimedean field then k° is its valuation ring with maximal ideal $k^{\circ\circ}$ and residue field $\widetilde{k}$.

2.3.3. *Description of points of* $\mathbf{A}^1_k$.

Definition 2.3.7. Let l be a non-archimedean k-field. Recall that $e_{l/k}$ is the cardinality of $|l^\times|/|k^\times|$ (may be infinite) and $f_{l/k} = [\widetilde{l} : \widetilde{k}]$. Transcendental analogs of these cardinals are $E_{l/k} = \mathrm{rank}_{\mathbf{Q}}(|l^\times|/|k^\times| \otimes_{\mathbf{Z}} \mathbf{Q})$ and $F_{l/k} = \mathrm{tr.deg.}(\widetilde{l}/\widetilde{k})$.

Exercise 2.3.8. Assume that l is algebraic over the completion of a subfield l_0 of transcendence degree n over k. Prove the Abhyankar inequality: $E_{l/k} + F_{l/k} \le n$.

We will use this result to classify points on $\mathbf{A}^1_k = \mathcal{M}(k[T])$ (and a similar argument classifies points on any k-analytic curve).

Definition/Exercise 2.3.9. (0) A point $x \in \mathbf{A}^1_k$ is *Zariski closed* if $|\ |_x$ has a non-trivial kernel. Show that this happens if and only if $\mathcal{H}(x)$ is finite over k. Show that otherwise $\mathcal{H}(x)$ is a completion of $k(T)$ and to give a not Zariski closed point is the same as to give a real valuation on $k(T)$ that extends that of k. In particular, $E_{\mathcal{H}(x)/k} + F_{\mathcal{H}(x)/k} \le 1$.

(1) x is of type 1 if $\mathcal{H}(x) \subset \widehat{k^a}$.

(2) x is of type 2 if $F_{\mathcal{H}(x)/k} = 1$.

(3) x is of type 3 if $E_{\mathcal{H}(x)/k} = 1$.

(4) x is of type 4 if $E_{\mathcal{H}(x)/k} = F_{\mathcal{H}(x)/k} = 0$ and x is not of type 1.

(5)* Show that in case (1) $\mathcal{H}(x)$ may contain an infinite algebraic extension of k (e.g. if $k = \mathbf{Q}_p$ it may coincide with $\mathbf{C}_p = \widehat{\mathbf{Q}}^a_p$). In particular, the map $\mathbf{A}^1_{\widehat{k^a}} \to \mathbf{A}^1_k$ usually has infinite (pro-finite) fibers. (Hint: fix elements $x_i \in k^s$ and take $T = \sum_{i=1}^\infty a_i x_i$ where $a_i \in k$ converge to zero fast enough; then use Krasner's lemma to show that $k(x_i) \subset \widehat{k(T)}$.)

Remark 2.3.10. More generally, $\mathcal{H}(x)$ may contain an infinite algebraic extension of k for type 4 points, but not for type 2 or 3 points.

Assume now that k is algebraically closed and let us describe the points of $\mathbf{A}^1_k$ in more details.

Exercise 2.3.11. Show that a valuation on $k[T]$ is determined by its values on the elements $T - a$ with $a \in k$. The number $r = \inf_{a\in k} |T - a|$ is called the *radius* of x (with respect to the fixed coordinate T).

(ii) Assume that the infimum r is achieved, say $r = |T - a|$. Show that

(a) if $r = 0$ then x is Zariski closed and of type 1.

(b) if $r > 0$ then x is the maximal point of the disc $E(a, r) = \mathcal{M}(k\{r^{-1}(T-a)\})$ of radius r and with center at a (i.e. $\sum_{i=0}^{n} a_i(T-a)^i = \max_i |a_i| r^i$). If r is *rational* in the sense that $r^n \in |k^\times|$ for some integral $n > 0$ then x is of type 2, and otherwise x is of type 3.

(iii) Assume that the infimum is not achieved, say $a_i \in k$ are such that the sequence $r_i = |T - a_i|$ decreases and tends to r. Then x is the only point in the intersection of the discs $E(a_i, r_i)$ and x is of type 4. In particular, type 4 points exist if and only if k is not *spherically complete*, i.e. there exist nested sequences of discs over k without common k-points.

Actually, $\mathbf{A}^1_k$ is a sort of an infinite tree whose leaves are type 1 and 4 points.

Exercise 2.3.12. (i) Use the previous exercise to prove that $\mathbf{A}^1_k$ is pathwise connected and simply connected. Moreover, show that for any pair of points $x, y \in \mathbf{A}^1_k$ there exists a unique path $[x, y]$ that connects them. (Hint: $[x, y] = [x, z] \cup [z, y]$ where z is the maximal point of the minimal disc containing both x and y and the open path (x, z) (resp. (z, y)) consists of the maximal points of discs that contain x but not y (resp. y but not x).)

(ii) Show that $\mathbf{A}^1_k \setminus \{x\}$ is connected whenever x is of type 1 or 4, consists of two components when x is of type 3, and consists of infinitely many components naturally parametrized by $\mathbf{P}^1_{\widetilde{k}}$ when x is of type 2. Thus, $\mathbf{A}^1_k$ is an infinite tree with infinite ramification at type 2 points. If k is trivially valued then there is just one type 2 point and no type 4 points, so the tree looks like a star whose rays connect the type 2 point (the trivial valuation) with the Zariski closed points.

3. Affinoid algebras and spaces

3.1. **Affinoid algebras.**

3.1.1. *The definition.*

Definition 3.1.1. (i) A *k-affinoid algebra* $\mathcal{A}$ is a Banach k-algebra that admits an admissible surjective homomorphism from a Banach algebra of the form $k\{\underline{r}^{-1}\underline{T}\}$. We say that $\mathcal{A}$ is *strictly k-affinoid* if one can choose $r_i \in |k^\times|$. More generally, we say that $\mathcal{A}$ is *H-strict* for a group $|k^\times| \subseteq H \subseteq \mathbf{R}^\times_+$ if one can choose such a homomorphism with $r_i \in H$.

(ii) The category of (resp. H-strict, resp. strictly) k-affinoid algebras with bounded morphisms is denoted *k-$AfAl$* (resp. *k_H-$AfAl$*, resp. *st-k-$AfAl$*). It will also be convenient to say *k_H-affinoid algebra* instead of H-strict k-affinoid algebra.

Exercise 3.1.2. Check that H-strictness depends only on the group $\sqrt{H}$ consisting of all elements $h^{1/n}$ with $h \in H$ and integral $n \geq 1$.

Remark 3.1.3. The group $\sqrt{H}$ is not dense in $\mathbf{R}^\times_+$ if and only $H = 1$ and 1-strict spaces are precisely the strictly analytic spaces over a trivially valued field. The case of $H = 1$ is degenerate and often demonstrates a very special behavior. We will ignore it in all cases when it requires a separate argument.

Example/Exercise 3.1.4. Let $\underline{r} = (r_1, \ldots, r_n)$ be a tuple of positive real numbers linearly independent over $|k^\times|$. Show that the k-affinoid ring

$$K_{\underline{r}} := k\{\underline{r}^{-1}\underline{T}, \underline{r}\ \underline{T}^{-1}\} = k\{\underline{r}^{-1}\underline{T}, \underline{r}\ \underline{S}\}/(T_1S_1 - 1, \ldots, T_nS_n - 1)$$

is a field and $K_{\underline{r}} \widetilde{\to} K_{r_1} \widehat{\otimes}_k K_{r_2} \widehat{\otimes}_k \dots \widehat{\otimes}_k K_{r_n}$.

3.1.2. *Basic properties.* Here is a summary of basic properties of k-affinoid algebras.

Fact 3.1.5. (i) Any affinoid algebra $\mathcal{A}$ is noetherian and all its ideal are closed.

(ii) If $f \in \mathcal{A}$ is not nilpotent then there exists $C > 0$ such that $\|f^n\| \le C\rho(f)^n$ for all $n \ge 1$. In particular, f is not quasi-nilpotent (i.e. $\rho(f) > 0$), and so ρ is a norm if and only if $\mathcal{A}$ is reduced.

(iii) If $\mathcal{A}$ is reduced then the Banach norm on $\mathcal{A}$ is equivalent to the spectral norm.

(iv) $\mathcal{A}$ is H-strict if and only if $\rho(\mathcal{A}) \subseteq \{0\} \cup \sqrt{H}$.

In particular, $(A/\mathrm{Ker}(\rho_A, \rho_A)$ is equivalent to quotient of A by its radical (provided with the residue semi-norm). In view of Remark 2.1.6, this can interpreted as equivalence of the "topological reduction" of A and the usual reduction of A. The following example shows that even naively looking k-Banach algebras do not have to satisfy the same nice conditions.

Example/Exercise 3.1.6. Let k be complete non-perfect field with a non-trivial valuation (e.g. $k = \mathbf{F}_p((t))$). Take any element x lying the completed perfection of k and non-algebraic over k (e.g. $x = t^{1+1/p} + t^{2+2/p^2} + \dots$) and let K be the closure of $k(x)$ in $\widehat{k^a}$. (Note that $K = \mathcal{H}(z)$ for a not Zariski closed point $z \in \mathbf{A}^1_k$ of type 1.) Show that the element $1 \otimes x - x \otimes 1$ is a quasi-nilpotent element of $K \widehat{\otimes}_k K$ which is not nilpotent.

For strictly affinoid algebras one can say more.

Fact 3.1.7. Let $\mathcal{A}$ be a strictly k-affinoid algebra (and the trivially values case is allowed).

(i) Noether normalization: there exists a finite admissible injective homomorphism $k\{T_1, \dots, T_n\} \to \mathcal{A}$.

(ii) Hilbert Nullstelensatz: $\mathcal{A}$ has a point in a finite extension of k.

(iii) The rings $k\{T_1, \dots, T_n\}$ are universally catenary of dimension n.

Remark 3.1.8. (i) The modern proof of Fact 3.1.5 is natural and systematic but rather long. First one develops Weierstrass theory (preparation and division theorems) for a strictly affinoid algebra $\mathcal{A}$. As a corollary one deduces analogs of two famous theorems about affine algebras: Noether normalization and Hilbert Nullstelensatz. All these results are used to establish Fact 3.1.5 in the strict case. For a non-strict $\mathcal{A}$ Fact 3.1.5 is deduced by descent, using the obvious fact that for any k-affinoid algebra $\mathcal{A}$ its base change $\mathcal{A} \widehat{\otimes}_k K_r$ is strictly K_r-affinoid for an appropriate K_r.

(ii) It was not studied in the literature whether one can develop this theory for all affinoid algebras. My expectations are as follows. Weierstrass theory can be developed for all affinoid algebras. Hilbert Nullstelensatz holds in a corrected form that any affinoid $\mathcal{A}$ has a point in a finite extension of some K_r. I expect that the following weak form of Noether normalization is the best one can get (see example 6.1.3(ii)). There exists injective homomorphisms $f : k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to \mathcal{A}'$ and $g : \mathcal{A}' \to \mathcal{A}$ such that f is finite admissible and g has dense image (then $\mathcal{M}(\mathcal{A})$ is a Weierstrass domain in a finite surjective covering $\mathcal{M}(\mathcal{A}')$ of a polydisc).

Fact 3.1.5 has the following corollary, which is very important when developing the theory of affinoid spaces.

Exercise 3.1.9. Assume that $\phi : \mathcal{A} \to \mathcal{B}$ is a bounded homomorphism of k-affinoid algebras, $f_1, \ldots, f_n \in \mathcal{B}$ are elements and $r_1, \ldots, r_n > 0$ are real numbers. Then ϕ extends to a bounded homomorphism $\psi : \mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B}$ with $\psi(T_i) = f_i$ if and only if $\rho_{\mathcal{B}}(f_i) \le r_i$.

Definition 3.1.10. Let $\mathcal{A}$ be a k-affinoid algebra. Any Banach $\mathcal{A}$-algebra that admits an admissible surjective homomorphism from $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$ is called $\mathcal{A}$-*affinoid*. Obviously, it is also a k-affinoid algebra.

3.1.3. *Finite $\mathcal{A}$-modules.* It turns out that the theory of finite Banach $\mathcal{A}$-modules is essentially equivalent to the theory of finite $\mathcal{A}$-modules.

Definition 3.1.11. A Banach $\mathcal{A}$-module M is *finite* if it admits an admissible surjective homomorphism from $\mathcal{A}^n$.

Fact 3.1.12. (i) The categories of finite Banach $\mathcal{A}$ modules and finite $\mathcal{A}$-modules are equivalent via the forgetful functor. In particular, any $\mathcal{A}$-linear map between finite Banach $\mathcal{A}$-modules is admissible.

(ii) Completed tensor product with a finite Banach $\mathcal{A}$-module M coincides with the usual tensor product. Namely, $M \otimes_{\mathcal{A}} N \widetilde{\to} M \widehat{\otimes}_{\mathcal{A}} N$ for any Banach $\mathcal{A}$-module N.

Exercise 3.1.13. Formulate and prove an analog of Fact 3.1.12 for the category of finite $\mathcal{A}$-algebras. In addition, prove that any finite Banach $\mathcal{A}$-algebra is $\mathcal{A}$-affinoid.

3.1.4. *Complements.*

Fact 3.1.14. (i) Fibred coproducts exist in the category k_H-$AfAl$ and coincide with completed tensor products.

(ii) For any non-archimedean k-field K, the correspondence $\mathcal{A} \mapsto \mathcal{A}\widehat{\otimes}_k K$ provides a ground field extension functor $k_H\text{-}AfAl \to K_{H|K^\times|}\text{-}AfAl$ compatible with completed tensor products.

Fact 3.1.15. The ground field k is not trivially valued if and only if any homomorphism between k-affinoid algebras is bounded.

The latter fact was known only for strictly affinoid algebras, so we suggest a proof below.

Exercise 3.1.16. (i) Show that any automorphism of K_r from Example 3.1.4 is bounded if and only if k is not trivially valued. (Hint: you have to use arithmetical properties of K_r because $\widehat{K_r^a}$ obviously has a lot of non-bounded automorphisms.)

(ii)* Prove Fact 3.1.15 in general. (Hint: use Shilov boundary from §3.4.1 to show that for a k-affinoid algebra $\mathcal{A}$ with an element f the spectral seminorm $\rho(f)$ can be described as the minimal number r such that for any $a \in k^a$ with $|a| > r$ the element $f + a \in \mathcal{A}\widehat{\otimes}_k k^a$ possesses a root of any natural degree prime to $\mathrm{char}(\widetilde{k})$.)

This fact is sometimes convenient but seems to be rather accidental. We will not use it; anyway, it does not hold for trivially valued ground fields. Here is one more example of an additional care trivially valued fields require to exercise; it shows that the class of finite and admissible homomorphisms of affinoid algebras is the right analog of the class of finite homomorphisms of affine algebras.

Exercise 3.1.17. (i) Show that a finite homomorphism is admissible whenever k is not trivially valued, but not in general. Also, give an example of a finite bounded homomorphism with a non-finite ground field extension. (Hint: $k[T] \to k[T]$ with different norms does the job.)

(ii)* Show that a homomorphism of k-affinoid algebras $\phi : \mathcal{A} \to \mathcal{B}$ is finite admissible if and only if its ground field extension $\phi\widehat{\otimes}_k K$ is finite and admissible. (Hint: the difficult case is to show the descent. First find $K_{\underline{r}}$ such that the algebras become strict over $K_{\underline{r}}$ and lift to the completion L of $\mathrm{Frac}(K \otimes_k K_{\underline{r}})$. The descent from $\phi\widehat{\otimes}_k L$ to $\phi\widehat{\otimes}_k K_{\underline{r}}$ is easy because everything is strictly affinoid, and the decent from $\phi\widehat{\otimes}_k K_{\underline{r}}$ to ϕ can be done by hands, since $K_{\underline{r}}$ has a nice explicit description.)

3.2. **Affinoid domains.**

3.2.1. *Affinoid spectra.* In the sequel we will develop a theory of k_H-analytic spaces builded from spectra of k_H-affinoid spaces (see [CT]). The two extreme choices of H will correspond to the general k-analytic spaces and strictly k-analytic spaces from [Ber2] and [Ber3]. So, from now on an intermediate group $|k^\times| \subseteq H \subseteq \mathbf{R}^\times$ is fixed, and if not said oppositely all k-analytic and k-affinoid spaces are assumed to be H-strict.

Remark 3.2.1. When the valuation on k is not trivial, it is important to develop the theory of strictly analytic spaces because it has connections to other approaches to non-archimedean geometry: formal geometry over k°, rigid geometry and adic geometry. We prefer to develop the general H-strict theory because it includes both the theory of general k-analytic spaces and the theory of strictly k-analytic spaces, and hence we do not have to distinguish these two cases in some formulations.

The category of *k_H-affinoid spectra* is a category dual to the category of k_H-affinoid algebras. Its objects are topological spaces of the form $\mathcal{M}(\mathcal{A})$ with a k_H-affinoid algebra $\mathcal{A}$ and morphisms are maps of the form $\mathcal{M}(f)$ for bounded homomorphisms $f : \mathcal{A} \to \mathcal{B}$. The k_H-affinoid spectra are objects of global nature that will later be enriched to more geometrical affinoid spaces. This will be done in the next two sections; we will localize the current construction by introducing an appropriate Grothendieck topology and structure sheaf.

3.2.2. *Generalized normed localization.* Topology on affine schemes is defined by localization. For a k_H-affinoid algebra $\mathcal{A}$ and an element $f \in \mathcal{A}$, the localization $\mathcal{A}_f$ is not affinoid for an obvious reason – we did not worry to extend the norm. The formula $\mathcal{A}_f = \mathcal{A}[T]/(Tf-1)$ leads to an idea to consider the k-affinoid algebras $\mathcal{A}_{r^{-1}f} = \mathcal{A}\{rT\}/(Tf-1)$. It turns out that the latter normed localization is not generic enough but its natural extension described below does the job. In the sequel, let $X = \mathcal{M}(\mathcal{A})$ be a k_H-affinoid spectrum.

Definition/Exercise 3.2.2. (i) Assume that elements $g, f_1, \dots, f_n \in \mathcal{A}$ do not have common zeros and $r_1, \dots, r_n \in \sqrt{H}$ are positive numbers. Show that

$$\mathcal{A}_V = \mathcal{A}\left\{\underline{r}^{-1}\frac{f}{g}\right\} := \mathcal{A}\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}/(gT_1 - f_1, \dots, gT_n - f_n)$$

is the universal $\mathcal{A}$-affinoid algebra such that $\rho_{\mathcal{A}_V}(f_i) \le r_i \rho_{\mathcal{A}_V}(g)$ for $1 \le i \le n$. Deduce that the map $\phi_V : \mathcal{M}(\mathcal{A}_V) \to X$ is a bijection onto

$$V = X\left\{\underline{r}^{-1}\frac{f}{g}\right\} := \{x \in X|\ |f_i(x)| \le r_i|g(x)|,\ 1 \le i \le n\}$$

Show that ϕ_V satisfies the following universal property: any morphism of k-affinoid spectra $\mathcal{M}(\mathcal{B}) \to X$ with image in V factors through $\mathcal{M}(\mathcal{A}_V)$. The compact subset V is called an H-strict *rational domain* in X and by the universal property one can identify it with the k_H-affinoid spectrum $\mathcal{M}(\mathcal{A}_V)$.

(ii) For any choice of $g_1, \dots, g_m, f_1, \dots, f_n \in \mathcal{A}$ and $s_1, \dots, s_m, r_1, \dots, r_n \in \sqrt{H}$ introduce analogous domains

$$V = X\{\underline{r}^{-1}\underline{f}, \underline{s}\ \underline{g}^{-1}\} = \{x \in X|\ |f_i(x)| \le r_i, |g_j(x)| \ge s_j\}$$

with algebras

$$\mathcal{A}_V = \mathcal{A}\{\underline{r}^{-1}\underline{f}, \underline{s}\ \underline{g}^{-1}\} = \mathcal{A}\{\underline{r}^{-1}\underline{T}, \underline{s}\ \underline{R}^{-1}\}/(T_i - f_i, g_j R_j - 1)$$

and prove the analogs of all properties from (i). These H-strict domains are called *Laurent* (resp. *Weierstrass* in the case when $m = 0$).

(iii) Show that any Laurent domain is rational and Laurent domains form a foundamental family of neighborhoods of a point whenever $H = 1$. Show that the latter is false for $H = 1$.

(iv) Show that $V \subseteq X$ is an H-strict rational, Laurent or Weierstrass domain if and only if it is a k-affinoid domain of the same type and the k-affinoid algebra $\mathcal{A}_V$ is H-strict. (Hint: use Fact 3.1.5.)

(v) For any map of k-affinoid spectra the preimage of rational, Laurent or Weierstrass domain is a domain of the same type and given by the same inequalities. In particular, all three classes of domains are closed under finite intersections.

(vi) Show that the classes of rational and Weierstrass domains are transitive (e.g. if Y is rational in X and Z is rational in Y then Z is rational in X), but Laurent domains are not transitive. Actually, this transitivity property is the main reason to extend the class of Laurent domains.

Although, we had to consider a more general type of localizations than in the theory of affine algebra, the main difference with the theory of varieties is that affinoid domains are compact and hence have to be closed. This fact will have serious consequences when we will develop the theory of coherent sheaves.

Example/Exercise 3.2.3. (i) Let $X = \mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$ be a polydisc with center at 0 and of polyradius $\underline{r}$, let $s_i \le r_i$ be positive numbers, and let $a_i \in k$ be elements with $|a_i| \le r_i$. Then the polydisc $\mathcal{M}(k\{s_1^{-1}(T_1 - a_1), \dots, s_n^{-1}(T_n - a_n)\})$ with center at $\underline{a}$ and of polyradius $\underline{s}$ is a Weierstrass domain in X.

(ii) For $s \le r$ the annulus $A(0; s, r) = \mathcal{M}(k\{r^{-1}T, sT^{-1}\})$ is a Laurent but not Weierstrass domain in the disc $E(0, r) = \mathcal{M}(k\{r^{-1}T\})$.

(iii) Any finite union of discs in $E(0, r)$ is a Weierstrass domain (and is a disjoint union of finitely many discs). In particular, even when $\mathcal{A}$ is an integral domain, its generalized localization does not have to be integral.

3.2.3. *General affinoid domains.* It is difficult to describe a general open affine subscheme explicitly but one can easily characterize it by a universal property. Here is an affinoid analog, which was already checked for rational domains in 3.2.2.

Definition 3.2.4. A closed subset $V \subset X$ is called a k_H-*affinoid domain* if there exists a morphism of k_H-affinoid spectra $\phi : \mathcal{M}(\mathcal{A}_V) \to X$ whose image is contained in V and such that any morphism of k_H-affinoid spectra $\mathcal{M}(\mathcal{B}) \to X$ with image in V factors through $\mathcal{M}(\mathcal{A}_V)$.

Note that $\mathcal{A}_V$ is unique up to a canonical isomorphism. The following fact allows us to identify V with the k_H-affinoid spectrum $\mathcal{M}(\mathcal{A}_V)$.

Fact 3.2.5. If $H \neq 1$ then non-empty fibers of ϕ are isomorphisms, i.e. ϕ is a bijection onto V and for any $y \in \mathcal{M}(\mathcal{A}_V)$ we have that $\mathcal{H}(\phi(y)) \widetilde{\to} \mathcal{H}(y)$.

Exercise 3.2.6. (i) Prove the Fact 3.2.5 for a point y with $[\mathcal{H}(y) : k] < \infty$.

(ii)* Prove Fact 3.2.5 in general. (Hint: first prove that $\mathcal{A}_V \widehat{\otimes}_{\mathcal{A}} \mathcal{A}_V \widetilde{\to} \mathcal{A}_V$ (in particular, the separated completion homomorphism $\mathcal{A}_V \otimes_{\mathcal{A}} \mathcal{A}_V \to \mathcal{A}_V \widehat{\otimes}_{\mathcal{A}} \mathcal{A}_V$ usually has a huge kernel); also, use without proof a non-trivial result of Gruson that the completion homomorphism $\mathcal{B} \otimes_k \mathcal{B} \to \mathcal{B} \widehat{\otimes}_k \mathcal{B}$ is injective for any k-Banach algebra $\mathcal{B}$.)

(iii) Show that V is Weierstrass if and only if the image of the homomorphism $\mathcal{A} \to \mathcal{A}_V$ is dense.

(iv) Show that fact 3.2.5 fails for $H = 1$.

Since Fact 3.2.5 is of fundamental importance, it does not make any sense to deal with the case of $H = 1$ in the sequel. From now on, $H \neq 1$. The following result shows that our definition of generalized localization was general enough. It allows to use rational domains for all local computations on affinoid spectra.

Fact 3.2.7 (Gerritzen-Grauert theorem)**.** Any k_H-affinoid domain in X is a finite union of H-strict rational domains in X.

Remark 3.2.8. This result (in the strict case) was not available in the first version of rigid geometry due to Tate. For this reason, Tate simply worked with rational domains and did not consider the general affinoid domains. In rigid geometry, the theorem was proved by Gerritzen-Grauert, and in Berkovich geometry the non-strict case was first deduced by Ducros. Two known rigid-theoretic proofs of this result are rather long and difficult. Originally, this theorem was needed to develop the very basics of analytic geometry, including Fact 3.2.5. Later it was shown in [Tem3] that Fact 3.2.5 can be proved independently, and then Gerritzen-Grauert theorem can be deduced rather shortly.

Exercise 3.2.9. Let X be a k_H-affinoid spectrum. Show that $V \subset X$ is a k_H-affinoid domain if and only if it is a k-affinoid domain with H-strict algebra $\mathcal{A}_V$ (Hint: use Gerritzen-Grauert theorem).

3.3. G-topology and the structure sheaf.

3.3.1. *G-topology of compact domains.* In order to define k-affinoid spaces we should provide each spectrum $X = \mathcal{M}(\mathcal{A})$ with a certain structure sheaf $\mathcal{O}_X$. Naturally, we would like $\mathcal{O}_X$ to be a sheaf of k-affinoid or k-Banach algebras but then we should study the sections over closed subsets (e.g. affinoid domains). A naive attempt to consider the topology generated by affinoid domains does not work out.

Exercise 3.3.1. Observe that the unit interval $[0, 1]$ is neither connected nor compact in the topology generated by closed intervals $[a, b]$. Show, similarly, that the unit closed disc $\mathcal{M}(k\{T\})$ is neither compact nor connected in the topology generated by affinoid domains.

A brilliant idea of Tate (with a strong influence of Grothendieck) is to generalize the notion of topology by allowing only certain open coverings. The resulting notion of a G-topology τ (which we prefer not to formulate with all details) is simply a Grothendieck topology on a set $\tau_{\rm op}$ of subsets of X such that $\tau_{\rm op}$ is closed under finite intersections and any covering of this topology is also a set-theoretical covering. Sets of $\tau_{\rm op}$ are called τ-open or G-open and the coverings of this Grothendieck topology are called *admissible coverings.* (Note that X is G-open because it is the intersection indexed by the empty set.)

Definition 3.3.2. (i) A *compact k_H-analytic domain* Y in a k_H-affinoid spectrum X is a finite union of k_H-affinoid domains. (It is called a special domain in [Ber1].)

(ii) The *H-strict compact G-topology* τ^c_H on a k_H-affinoid spectrum X has compact k_H-analytic domains as open sets and finite coverings as admissible coverings.

Remark 3.3.3. By Gerritzen-Grauert theorem one can replace affinoid domains with rational domains in this definition obtaining the original Tate's definition of G-topology.

3.3.2. *The structure sheaf.* Tate proved that (in the strict case) this G-topology is the right tool to define coherent sheaves of modules. In particular, $\mathcal{O}_{X_H}(V)=\mathcal{A}_V$ extends to a τ^c_H-sheaf of Banach algebras.

Fact 3.3.4 (Tate's acyclity theorem)**.** For any finite affinoid covering $X=\cup_i V_i$ and finite Banach $\mathcal{A}$-module M the Čhech complex

$$0\to M\to\prod_i M_i\to\prod_{i,j}M_{ij}\to\dots$$

is exact and admissible, where $M_i=M\otimes_{\mathcal{A}}\mathcal{A}_{V_i}$, $M_{ij}=M\otimes_{\mathcal{A}}\mathcal{A}_{V_{ij}}$, etc.

Admissibility in this result is very important. In particular, it allows to define norms on the structure sheaf introduced below.

Exercise 3.3.5. (i) For any compact k_H-analytic domain V with a finite affinoid covering $V=\cup_i V_i$ set $\mathcal{O}_{X_H}(V)=\mathrm{Ker}(\prod_i\mathcal{A}_{V_i}\to\prod_{i,j}\mathcal{A}_{V_i\cap V_j})$ and provide it with the restriction norm. Show that $\mathcal{O}_{X_H}(V)$ depends only on V, and $\mathcal{O}_{X_H}$ is a sheaf of k-Banach algebras. In particular, the restriction morphisms are bounded.

(ii) Deduce that any polydisc $X=\mathcal{M}(k\{r_1^{-1}T_1,\dots,r_n^{-1}T_n\})$ with $r_i\in\sqrt{H}$ is τ^c_H-connected, i.e. X is not a disjoint union of two non-empty compact k-analytic domains. (Hint: $\mathcal{O}_{X_H}(X)$ is integral.)

3.3.3. *k_H-affinoid spaces.*

Definition 3.3.6. (i) A *k_H-affinoid space* X is a k_H-affinoid spectrum $\mathcal{M}(\mathcal{A})$ provided with the G-topology τ^c_H and the τ^c_H-sheaf of k-Banach rings $\mathcal{O}_{X_H}$, which is called the *structure sheaf.*

(ii) A *morphism* of k_H-affinoid spaces is a continuous and τ^c_H-continuous map $f:Y\to X$ provided with a *bounded* homomorphism of sheaves $f^\#:\mathcal{O}_{X_H}\to f_*(\mathcal{O}_{Y_H})$ in the sense that for any pair of compact k_H-analytic domains $X'\subset X$ and $Y'\subset f^{-1}(X)$ the homomorphism $f^\#:\mathcal{O}_{X_H}(X')\to\mathcal{O}_{Y_H}(Y')$ is bounded.

(iii) A τ^c_H-sheaf of finite $\mathcal{O}_{X_H}$-Banach modules is *coherent* if it is of the form $\mathcal{M}(V)=M\otimes_{\mathcal{A}}\mathcal{O}_{X_H}(V)$ for a finite Banach $\mathcal{A}$-module M.

Exercise 3.3.7. Show that the continuity assumption in (ii) follows from τ^c_H-continuity and hence can be removed.

Note that k_H-affinoid spaces are "self-contained" geometric objects analogous to affine schemes. It will later be an easy task to globalize this definition.

Fact 3.3.8. The categories of k_H-affinoid spectra and k_H-affinoid spaces are naturally equivalent.

Since this fact seems to be new, we give a detailed exercise on its proof.

Exercise 3.3.9. (i) Reduce the question to the following statement: if $(f, f^{\#}) : (Y, \mathcal{O}_{Y_H}) \to (X, \mathcal{O}_{X_H})$ is a morphism of k_H-affinoid spaces and $\phi : \mathcal{O}_{X_H}(X) \to \mathcal{O}_{Y_H}(Y)$ is the induced bounded homomorphism of k_H-affinoid algebras, then $f = \mathcal{M}(\phi)$ and $f^{\#} : \mathcal{O}_{X_H} \to f_*(\mathcal{O}_{X_H})$ is the bounded homomorphism of the structure sheaves induced by ϕ. In other words, $(f, f^{\#})$ is, in its turn, induced by ϕ.

(ii) Choose a point $y \in Y$ with $x = f(y)$. For any H-strict rational domain $X' = X\{r^{-1}\frac{g}{h}\}$ containing x choose an H-strict rational domain $Y' \subseteq f^{-1}(X')$ containing y. Set $\mathcal{A} = \mathcal{O}_{X_H}(X)$, $\mathcal{B} = \mathcal{O}_{Y_H}(Y)$ and $\mathcal{B}' = \mathcal{O}_{Y_H}(Y')$. Since the homomorphism $\mathcal{A} \to \mathcal{B}'$ factors through $\mathcal{O}_{X_H}(X') = \mathcal{A}\{r^{-1}\frac{g}{h}\}$, it follows that the homomorphism $\mathcal{B} \to \mathcal{B}'$ factors through $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{A}\{r^{-1}\frac{g}{h}\} \widetilde{\to} \mathcal{B}\{r^{-1}\frac{g}{h}\}$, and hence $Y' \subseteq Y\{r^{-1}\frac{g}{h}\}$. Therefore, any inequality $|g(x)| \le r|h(x)|$ with $g, h \in \mathcal{O}_{X_H}(X)$ implies that $|\phi(g)(y)| \le r|\phi(h)(y)|$. Deduce that $\mathcal{M}(\phi)$ takes y to x and hence coincides with f.

(iii) Finish the argument by showing that $f^{\#}$ is also induced by ϕ. (Hint: check this for sections on rational domains and then apply Tate's theorem.)

In the sequel we will not distinguish between k_H-affinoid spectra and k_H-affinoid spaces, that is, we will automatically enrich any k_H-affinoid spectrum to the structure of a k_H-affinoid space. Also, we will refine the structure of k_H-affinoid spaces a little bit more in §4.1.

3.3.4. *Coherent sheaves.* We finish §3.3 with a discussion on coherent sheaves. By Tate's theorem any finite Banach $\mathcal{A}$-module gives rise to a sheaf of finite Banach $\mathcal{O}_{X_H}$-modules. The opposite result (in a slightly different formulation) was proved by Kiehl.

Fact 3.3.10. (i) (Kiehl's theorem) Any G-locally coherent sheaf is coherent. Namely, if for a finite Banach $\mathcal{O}_{X_H}$-module $\mathcal{M}$ there exists a finite affinoid covering $X = \cup_i V_i$ such that the restrictions $\mathcal{M}|_{V_i}$ are coherent then $\mathcal{M}$ is coherent.

(ii) Tate's and Kiehl's theorems easily imply that the categories of coherent $\mathcal{O}_{X_H}$-module and finite Banach $\mathcal{A}$-modules are naturally equivalent.

Remark 3.3.11. The theory of k_H-analytic (and even k_H-affinoid) spaces does not have a theory of infinite type modules analogous to the theory of quasi-coherent modules on schemes. Moreover, this theory does not even have a notion of affinoid morphisms. There exist morphisms $f : Y \to X$ with finite affinoid coverings $X = \cup_i X_i$ such that X and $f^{-1}(X_i)$ are affinoid but Y is not affinoid. The first such example is due to Q. Liu, see [Liu], and a simpler example can be found in [CT].

3.4. **The reduction map, boundary and interior.**

3.4.1. *Reduction.* In general, reduction relates (strictly) k-affinoid algebras and spaces to geometry over the residue field $\widetilde{k}$.

Fact 3.4.1. Assume that the valuation is non-trivial. For a bounded homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ of strictly k-affinoid algebras the following conditions are equivalent: ϕ is finite, $\widetilde{\phi} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is finite, $\phi^\circ : \mathcal{A}^\circ \to \mathcal{B}^\circ$ is integral.

Exercise 3.4.2. Deduce that for any strictly k-affinoid $\mathcal{A}$ the reduction $\widetilde{\mathcal{A}}$ is finitely generated over $\widetilde{k}$.

Remark 3.4.3. (i) This result shows that the reduction functor controls strictly k-affinoid algebras very well. A similar result holds for general k_H-affinoid algebras if one replaces $\widetilde{A}$ with the H-graded reduction

$$\widetilde{A}_H = \oplus_{h \in H} \{x \in \mathcal{A} | \ \rho(x) \leq h\} / \{x \in \mathcal{A} | \ \rho(x) < h\}$$

(ii) The question whether ϕ° is finite is more subtle. Already for a finite field extension l/k one often has that l°/k° is not finite. On the other hand if K is algebraically closed or discrete valued then ϕ° is integral if and only if it is finite.

Now, let us study the geometric side of the reduction.

Definition 3.4.4. (i) The *reduction* of a strictly k-affinoid space $X = \mathcal{M}(\mathcal{A})$ is the reduced $\widetilde{k}$-variety $\widetilde{X} = \mathrm{Spec}(\widetilde{\mathcal{A}})$.

(ii) The reduction map $\pi_X : X \to \widetilde{X}$ sends a point x with the character $\chi_x : \mathcal{A} \to \mathcal{H}(x)$ to the point $\widetilde{x} \in \widetilde{X}$ induced by the character $\widetilde{\chi}_x : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{H}(x)}$. (Note that $k(\widetilde{x})$ can be much smaller than $\widetilde{\mathcal{H}(x)}$; as we know from Exercise 2.3.9(5) the latter field does not even have to be finitely generated over $\widetilde{k}$.)

Exercise 3.4.5. The map π_X is *anti-continuous* in the sense that the preimage of open set is closed and vice versa.

Fact 3.4.6. (i) The reduction map is surjective.

(ii) The preimage of a generic point of $\widetilde{X}$ is a single point, and the union of all such points is the *Shilov boundary* $\Gamma(X)$ of X. Namely, any function $|f(x)|$ with $f \in \mathcal{A}$ accepts its maximum on $\Gamma(X)$ and $\Gamma(X)$ is the minimal closed set satisfying this property.

Remark 3.4.7. The same result holds for H-graded reduction if one defines $\widetilde{X}_H = \mathrm{Spec}_H(\widetilde{\mathcal{A}}_H)$ as the set of all homogeneous prime ideals in the H-graded ring $\widetilde{\mathcal{A}}_H$.

Example 3.4.8. (i) The spectral seminorm on $\mathcal{A}$ is multiplicative if and only if $\widetilde{\mathcal{A}}$ is integral. In this case, the spectral seminorm itself defines a point which is both the preimage of the generic point of $\widetilde{X}$ and the Shilov boundary of X. For example, the Shilov boundary of a polydisc $E(0, \underline{r}) = \mathcal{M}(k\{\underline{r}^{-1}\underline{T}\})$ is a single (maximal) point.

(ii) The Shilov boundary of an annulus $X = A(0; s, r) = E(0, r) \setminus E(0, s) = \mathcal{M}(k\{r^{-1}T, sT^{-1}\})$ with $s < r$ consists of two points, the maximal points of the discs $E(0, r)$ and $E(0, s)$. The reduction $\widetilde{X}$ is the cross $\mathrm{Spec}(\widetilde{k}[R, S]/(RS))$, where R and S are the reductions of appropriate rescalings of T and T^{-1}. What happens when $r = s$?

3.4.2. *The relative boundary and interior.*

Definition 3.4.9. Let $\phi : Y \to X$ be a morphism of k-analytic spaces and let $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$. The *relative interior* $\mathrm{Int}(Y/X) \subseteq Y$ consists of points $y \in Y$ such that there exists an admissible surjective $\mathcal{A}$-homomorphism $\psi : \mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B}$ with $|\psi(T_i)(y)| < r_i$ for $1 \leq i \leq n$. The *relative boundary* $\partial(Y/X)$ is the complement to the relative interior in Y. The absolute *interior* $\mathrm{Int}(Y)$ and *boundary* $\partial(Y)$ are defined with respect to the morphism $Y \to \mathcal{M}(k)$.

Remark 3.4.10. This definition has a very geometric interpretation as follows. The homomorphism ψ induces a closed immersion of Y into a relative closed polydisc of polyradius $\underline{r}$ over X. The inequalities in the definition imply that the image of y lies in the open relative polydisc of the same polyradius.

The notions of relative boundary and interior turn out to be very important in analytic geometry. Most of the facts about them are proved by use of the reduction theory. Here is a list of their basic properties.

Fact 3.4.11. (i) The relative interior is open in Y and the relative boundary is closed.

(ii) The relative interiors are compatible with compositions in the sense that $\mathrm{Int}(Z/X) = \mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X))$ for a pair of morphism $Z \xrightarrow{\psi} Y \xrightarrow{\phi} X$.

(iii) Relative boundary is G-local on the base in the sense that for a finite affinoid covering $X = \cup_i X_i$ and $Y_i = \phi^{-1}(X_i)$ one has that $\partial(Y/X) = \cup_i \partial(Y_i/X_i)$.

(iv) ϕ has an empty boundary if and only if it is *finite*, i.e. $\mathcal{A} \to \mathcal{B}$ is finite admissible.

(v) If Y is an affinoid domain in X then $\mathrm{Int}(Y/X)$ is the topological interior of Y in X.

(vi) Assume that X and Y are strictly k-affinoid, $y \in Y$ and $x = \phi(y)$. Then $y \in \mathrm{Int}(Y/X)$ if and only if the image of $\mathcal{B}$ in $k(\widetilde{y})$ (usually denoted $\widetilde{\chi}_y(\widetilde{\mathcal{B}})$) is finite over the image of $\mathcal{A}$ in $k(\widetilde{x})$.

Remark 3.4.12. The last condition is very convenient for explicit computations. Its analog holds for H-strict X and Y and H-graded reduction.

Now, we illustrate the introduced notions with some examples. For simplicity, all analytic spaces in the examples are assumed to be strict.

Example/Exercise 3.4.13. (i) Show that $\mathrm{Int}(Y)$ is the preimage under the reduction map π_Y of the set of closed points of $\widetilde{Y}$.

(ii) Let X be an *affinoid curve*, i.e. a k-affinoid space of dimension one (in the sense of §3.5 below) and without isolated Zariski closed points. Show that the boundary of X coincides with its Shilov boundary. (Hint: use Noether normalization to prove that $\widetilde{Y}$ is a curve.)

Next, let us consider a higher dimensional example.

Example/Exercise 3.4.14. Let ϕ be the projection of the unit polydisc $Y = \mathcal{M}(k\{T, S\})$ onto the unit disc $X = \mathcal{M}(k\{S\})$.

(i) For any $x \in X$ which is not the maximal point, the fiber $Y_x = \psi^{-1}(x)$ contains exactly one point of $\partial(Y/X)$ (the maximal point of the fiber disc).

(ii) If x is the maximal point of X then the intersection of Y_x with $\partial(Y/X)$ is huge. Let η and ε be the generic points of the quadrics $\widetilde{T}^2 - \widetilde{S} = 0$ and $\widetilde{T}\widetilde{S} = 1$ in $\widetilde{Y} = \mathrm{Spec}(\widetilde{k}[\widetilde{T}, \widetilde{S}])$. Show that $\pi_Y^{-1}(\eta) \subset \mathrm{Int}(Y/X)$ but $\pi_Y^{-1}(\varepsilon) \subset \partial(Y/X)$. Find a geometric explanation for this fact. (Hint: how these fibers are embedded into a larger polydisc $\mathcal{M}(k\{r^{-1}T, S\})$ with $r > 1$?)

(iii) Show that $\partial(Y)$ is strictly larger than $\partial(Y/X)$. Moreover, show that $\partial(Y) \cap Y_x$ contains subdiscs for infinitely many points $x \in X$, and $\partial(Y) \cap Y_x$ is the maximal point of Y_x for infinitely many points $x \in X$.

3.5. **The dimension theory.**

Exercise 3.5.1. (i) Prove that the dimension of a strictly k-affinoid algebra $\mathcal{A}$ is preserved under any ground field extension. (Hint: use Noether normalization.)

(ii) Show that general k-affinoid algebras do not share this property. (Hint: show that $K_r\widehat{\otimes}_k K_r \widetilde{\to} K_r\{r^{-1}T\} \widetilde{\to} K_r\{T\}$.)

Since the dimension stabilizes after a sufficiently large ground field extension, it is natural to define the dimension of k-affinoid spaces as follows.

Definition 3.5.2. Dimension $\dim(X)$ of a k-affinoid space $X = \mathcal{M}(\mathcal{A})$ is the dimension of an algebra $\mathcal{A}_K = \mathcal{A}\widehat{\otimes}_k K$, where K is a non-archimedean k-field such that $\mathcal{A}_K$ is strictly K-affinoid.

Remark 3.5.3. Assume for concreteness that $r = (r_1)$. One can view $X = \mathcal{M}(K_r)$ as a curve consisting of its generic point. The only difference with the theory of algebraic k-curves is that X is of "finite type" over k.

The following exercise illustrates that type 2, 3 and 4 points are sort of "generic points" of the curves, so (non-formally) one can imagine them as points of dimension 1.

Exercise 3.5.4. Let x be a point of $\mathbf{A}^1_k$.

(i) Show that x is of type 1 if and only if the fiber over x in any ground field extension $\mathbf{A}^1_K$ is profinite.

(ii) Show that x is not of type 1 if and only if the fiber over x in some $\mathbf{A}^1_K$ contains a closed K-disc.

4. Analytic spaces

4.1. **The category of k-analytic spaces.**

4.1.1. *Nets.* We would like general analytic spaces to be locally compact but not necessarily Hausdorff topological spaces covered by (or glued from) its affinoid domains. The intersections of affinoid domains do not have to be affinoid or even compact (in the non-Hausdorff case), but we still want this intersection to be an analytic space. In particular, it should be locally compact and covered by affinoid domains. This shows that in order to define an analytic space by an affinoid atlas we must act with care. The following notion axiomatizes the properties such atlas should satisfy.

Definition 4.1.1. Let X be a topological space with a set of subsets T.

(i) T is a *quasi-net* if any point $x \in X$ has a neighborhood of the form $\cup_{i=1}^n V_i$ with $x \in V_i \in T$ for $1 \le i \le n$.

(ii) a quasi-net T is a *net* if for any choice of $U, V \in T$ the restriction $T|_{U\cap V} = \{W \in T |\ W \subseteq U \cap V\}$ is a quasi-net.

Remark 4.1.2. This definition should not be confused with the nets that generalize sequences in the definition of limits. In a sense, they are analogous to ε-nets on metric spaces.

In our situation X is locally Hausdorff and the sets of T are compact, so let us assume that this is the case.

Exercise 4.1.3. (i) With the above assumptions, X is locally compact.

(ii) A subset $Y \subseteq X$ is open if and only if $Y \cap U$ is open in U for any $U \in T$.

(iii) X is Hausdorff if and only if for any pair $U, V \in T$ the intersection $U \cap V$ is compact.

(iv) A subset $Y \subseteq X$ is compact if and only if it is covered by compact intersections $Y \cap U$ with $U \in T$ (but there may exist non-compact intersections).

4.1.2. *Analytic spaces.* We will freely view a net as a category with morphisms being the inclusions.

Definition 4.1.4. A k_H*-analytic space* is a locally Hausdorff topological space X with an *atlas of* k_H*-affinoid domains*, which is a net τ_0 on X, a functor $\phi : \tau_0 \to k_H\text{-}Aff$ and an isomorphism of the two natural topological realization functors from τ_0 to the category of topological spaces Top and $i : Top \circ \phi$. In concrete terms, we will write $\phi(U) = \mathcal{M}(\mathcal{A}_U)$ and i reduces to giving a homeomorphism $i_U : U \widetilde{\to} \phi(U)$ for any $U \in \tau_0$ and such that for any inclusion $j : U \hookrightarrow V$ in τ these homeomorphisms are compatible with $\phi(j) : \phi(U) \to \phi(V)$.

By Exercise 4.1.3(i) X is locally compact.

Definition 4.1.5. (i) A k_H-analytic domain in X is a subset $V \subset X$ that admits a locally finite covering $V = \cup_{i \in I} V_i$ such that each V_i is a k_H-affinoid domain in some element of τ_0.

(ii) By τ_H (resp. τ_H^c) we denote the sets of all (resp. compact) k_H-analytic domains. A covering of an element of τ_H by elements of τ_H is *admissible* if it admits a locally finite refinement.

Exercise 4.1.6. (i) Show that τ_H with admissible coverings is a G-topology and give an example when it is not closed under finite unions. (Hint: already the union of open polydisc of polyradius $(1, 2)$ with the closed polydisc of polyradius $(2, 1)$ is not locally compact.)

(ii) Show that although τ_H^c does not have fibred products in general (e.g. X is not in τ_H^c if it is not compact and τ_H^c does not have to be closed under intersections of pairs for a non-Hausdorff space), it is closed under fibred products (i.e. if $U, V \subset W$ are in τ_H^c then $U \cap V$ is in τ_H^c). Use this to define τ_H^c-sheaves. In the sequel we will refer to τ_H as the *G-topology of* X.

(iii) Show that any admissible covering of a compact k_H-analytic domain by compact k_H-analytic domains possesses a finite subcovering. Deduce that $V \in \tau_H^c$ if and only if it $V = \cup_{i=1}^n V_i$ with each V_i being affinoid in an element of τ_0 and all intersections $V_i \cap V_j$ being compact.

(iv) Deduce that the correspondence $V \to \mathcal{A}_V$ on τ_0 extends uniquely to a τ_H^c-sheaf of Banach k-algebras.

The sheaf from (iv) will be called the *structure sheaf* and denoted $\mathcal{O}_{X_H}$. Any k_H-analytic space given by an atlas will automatically be provided with this additional structure. Now we can define what does it mean for two analytic spaces to be isomorphic.

Definition 4.1.7. An isomorphism between two analytic spaces X and X' is a homeomorphism $f : X' \to X$ which induces a bijection $\tau_{H,X'} \widetilde{\to} \tau_{H,X}$ and an isomorphism of the structure sheaves $f^{\#} : \mathcal{O}_{X_H} \to \mathcal{O}_{X'_H}$.

Note that we do not need to add any condition on compatibility of f and $f^{\#}$ thanks to Fact 3.3.8. If not said oppositely, we will consider k_H-analytic spaces without any fixed affinoid atlas.

4.1.3. *Morphisms between analytic spaces.* Intuitively, a morphism between k_H-analytic spaces should be a continuous and G-continuous map $f : Y \to X$ (i.e. the preimage of an analytic domain is an analytic domain) that is naturally compatible with atlases provided with a bounded homomorphism $f^{\#} : \mathcal{O}_{X_H} \to f_*\mathcal{O}_{Y_H}$. However, the direct image $f_*\mathcal{O}_{Y_H}$ does not really makes sense for non-compact morphisms, including $\frac{\mathbf{1}}{\mathbf{k}} \to \mathcal{M}(\mathbf{k})$. Therefore, we suggest the following definition.

Definition 4.1.8. A morphism $f : Y \to X$ between k_H-analytic spaces consists of a continuous and G-continuous map $f : Y \to X$ and a compatible family of bounded homomorphisms $f^{\#}_{U,V} : \mathcal{O}_{X_H}(U) \to \mathcal{O}_{Y_H}(V)$ for any pair of compact k_H-analytic domains $U \subseteq X$ and $V \subseteq f^{-1}(U)$.

A priori it is not clear how to compose such morphisms (for example, the image of a compact set does not have to be Hausdorff). This forces us to show that a morphism is determined already by its restriction to atlases. (Note that the atlas definition of morphisms is used in [Ber2, §1.2] and [Ber3, 3.1].)

Exercise 4.1.9. (i) Assume that Y and X are provided with affinoid atlases τ_Y and τ_X. Then to give a morphism $Y \to X$ is equivalent to give a similar data $(g, g^{\#})$ but with $g_{U,V}$ defined only for $U \subset \tau_X$ and $V \subset \tau_Y$ with $f(V) \subseteq U$.

(ii) Use this to define composition of morphisms.

(iii) Show that any k_H-analytic domain $V \subseteq X$ possesses a canonical structure of a k_H-analytic space. Moreover, the inclusion underlies the *analytic domain embedding* morphism $i_V : V \to X$ which possesses the universal property that any morphism $Y \to X$ with set-theoretical image in V factors through V (in the analytic category).

Thanks to the claim of (ii) we have now introduced the category of k_H-analytic spaces, which will be denoted k_H-*An*. The particular case of (resp. *strictly*) *k-analytic spaces* corresponding to $H = \mathbf{R}_+^\times$ (resp. $H = |k^\times|$) will be denoted *k-An* (resp. *st-k-An*). The following useful result is surprisingly difficult (for non-separated spaces). It can be interpreted as follows: if two analytic spaces are H-strict then any morphism between them can be described using H-strict atlases.

Fact 4.1.10. Assume that $H' \subseteq \mathbf{R}_+^\times$ is a subgroup containing H. The natural embedding functor k_H-*An* $\to k_{H'}$-*An* is fully faithful. In particular, any analytic space admits at most one (up to an isomorphism) structure of an H-strict analytic space.

In particular, even when studying k_H-analytic spaces we can (and in the sequel will) safely work with the category of all k-analytic spaces. In the sequel, our default G-topology is the G-topology of all k-analytic domains.

Exercise 4.1.11. Show that the τ^c-sheaf $\mathcal{O}_{X_{\mathbf{R}_+^\times}}$ uniquely extends to a G-sheaf of rings $\mathcal{O}_{X_G}$ and, moreover, $\mathcal{O}_{X_G}(V) = \mathrm{Mor}_{k-An}(V, \mathbf{A}_k^1)$.

Note that $\mathcal{O}_{X_G}$ is not a sheaf of Banach rings. From now on, *structure sheaf* refers to $\mathcal{O}_{X_G}$.

4.1.4. *Gluing of analytic spaces.* There are three main constructions of new k-analytic spaces: by gluing (or using atlases), by analytification of algebraic k-varieties, and as the generic fiber of formal k°-schemes. Here we consider only the first construction because the other two will be studied later.

Exercise 4.1.12. Assume that $\{X_i\}_{i\in I}$ is a family of k-analytic spaces provided with analytic domains $X_{ij} \hookrightarrow X_i$ and isomorphisms $\phi_{ij} : X_{ij} \widetilde{\to} X_{ji}$ that satisfy the usual cocyle compatibility condition on the intersections $X_{ijk} = X_{ij} \cap X_{ik}$. Show that in the following cases they glue to a k-analytic space X covered by domains isomorphic to X_i so that $X_i \cap X_j \widetilde{\to} X_{ij}$.

Case 1: The domains $X_{ij} \subseteq X_i$ are open.

Case 2; The domains $X_{ij} \subseteq X_i$ are closed and for each i only finitely many domains X_{ij} are non-empty.

Exercise 4.1.13. (i) Let X be glued from annuli $X_1 = A(0; r, s)$ and $X_2 = A(0; s, t)$ along $X_{12} = A(0; s, s)$ so that the orientation of the annuli is preserved. The latter means that the gluing homomorphism $k\{s^{-1}T_1, rT_1^{-1}\} \to k\{s^{-1}T, sT^{-1}\}$ takes T_1 to an element $\sum_{i=-\infty}^{\infty} a_i T^i$ with $|a_1 T| = s > |a_i T^i|$ for $i \neq 1$, and similarly for the second chart. Show that X is isomorphic to the annulus $A(0; r, t)$. (Hint: show that the intersection of $k\{s^{-1}T_1, rT_1^{-1}\}$ and $k\{t^{-1}T_2, sT_2^{-1}\}$ inside $k\{s^{-1}T, sT^{-1}\}$ is isomorphic to $k\{t^{-1}R, rR^{-1}\}$.)

(ii) In the same way show that if X_1 is the disc $E(0, s)$ then $X \widetilde{\to} E(0, t)$.

(iii) We define $\mathbf{P}^1_k$ as the obvious gluing of $\mathcal{M}(k\{T\})$ and $\mathcal{M}(k\{T^{-1}\})$ along $\mathcal{M}(k\{T, T^{-1}\})$. Show that any other gluing of two discs with the same choice of orientation is isomorphic to $\mathbf{P}^1_k$. A wrong choice of orientation leads to a space that we call closed unit disc with doubled open unit disc. This space is Hausdorff, but we will later see that it is not locally separated at the maximal point of the disc.

(iv) Define $\mathbf{P}^n_k$ with homogeneous coordinates $T_0, \ldots, T_n$ in two different ways: (i) as a gluing of $n+1$ unit polydiscs, (ii) as a gluing of $n+1$ affine spaces $\mathbf{P}^n_k$. (Hint: in both cases, it is convenient to symbolically denote coordinates on the i-th chart as $\frac{T_j}{T_i}$ for $0 \leq j \leq n, j \neq i$.)

Definition 4.1.14. (i) A seminorm on a graded ring $A = \oplus_{d\in\mathbf{N}} A_d$ is *homogeneous* if it is determined by its values on the homogeneous elements via the max formula $|\sum_{d\in\mathbf{N}} a_d| = \max_{d\in\mathbf{N}} |a_d|$.

(ii) Seminorms $|\ |$ and $\|\ \|$ on A are *homothetic* if there exists a number $C > 0$ such that $|a_d| = C^d \|a_d\|$ for any $a_d \in A_d$.

(iii) Let k be a non-archimedean field with a graded k-algebra A. The projective spectrum $\mathbf{P}\mathcal{M}(A)$ is the set of all homothety equivalence classes of homogeneous semivaluations on A that extend the valuation of k and do not vanish on the whole A_1.

Exercise 4.1.15. (i) Show that $\mathbf{P}^n_k = \mathbf{P}\mathcal{M}(k[T_0, \ldots, T_n])$ by a direct computation.

(ii) Alternatively, show that two points $x, y \in \mathbf{A}^{n+1} \setminus \{0\} = \mathcal{M}(k[\underline{T}]) \setminus \{0\}$ are mapped to the same point by the projection $\mathbf{A}^{n+1} \setminus \{0\} \to \mathbf{P}^n_k$ if and only if there exists $C > 0$ such that $|f_d(x)| = C^d |f_d(y)|$ for any homogeneous $f_d(\underline{T}) \in k[\underline{T}]$ of degree d. Show that the set of homogeneous semivaluations in any fiber of the projection is a single homothety equivalence class.

4.1.5. *Products and fibers of morphisms.*

Fact 4.1.16. (i) The category k_H-*An* possesses a fibred product $Y \times_X Z$ which agrees with the fibred product in any category $k_{H'}$-*An* for $H \subseteq H'$ and in the category of k-affinoid spaces. In particular, if $X = \mathcal{M}(A)$, $Y = \mathcal{M}(\mathcal{B})$ and $Z = \mathcal{M}(\mathcal{C})$ then $\mathcal{M}(\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{C})\widetilde{\to}Y \times_X Z$

(ii) Let $f : Y \to X$ and $g : Z \to X$ be morphisms of k-analytic spaces, and assume that $X = \cup_i(X_i)$, $f^{-1}(X_i) = \cup_j Y_{ij}$ and $f^{-1}(X_i) = \cup_k Z_{ik}$ are locally finite coverings by affinoid domains. Then $Y \times_X Z$ is covered by analytic domains $Y_{ij} \times_{X_i} Z_{ik}$.

Actually, the second part of this result indicates how the fibred product is constructed.

Definition/Exercise 4.1.17. Assume that $f : Y \to X$ is a morphism of k-analytic spaces and $x \in X$ is a point. Define the fibred product $Y_x = Y \times_X \mathcal{M}(\mathcal{H}(x))$ as an $\mathcal{H}(x)$-analytic space. Show that set-theoretically, Y_x is the preimage of x. The space Y_x is called the *fiber* of f over x.

4.2. **Basic classes of analytic spaces and morphisms.**

4.2.1. *Good spaces.*

Definition 4.2.1. (i) An analytic space X is called good if any its point possesses an affinoid neighborhood.

(ii) The sheaf $\mathcal{O}_X$ is defined as the restriction of $\mathcal{O}_{X_H}$ to the usual topology of X.

(iii) For any point $x \in X$ we define $\kappa(x)$ as the residue field of the local ring $\mathcal{O}_{X,x}$.

Good spaces are often more convenient to work with. In particular, the definition of $\mathcal{O}_X$ applies to any X, but it can play the role of a structure sheaf only for the class of good spaces (see Example 4.2.3). That is why we will only consider $\mathcal{O}_X$ on good spaces. In such case, $\mathcal{H}(X)$ is the completion of $\kappa(x)$.

Fact 4.2.2. For a good k_H-analytic space X the category of coherent $\mathcal{O}_{X_H}$-modules is equivalent to the category of *coherent* $\mathcal{O}_X$-modules (i.e. $\mathcal{O}_X$-modules locally isomorphic to a quotient of $\mathcal{O}_X^n$).

Example/Exercise 4.2.3. Let X be a closed unit disc with doubled open unit disc as in Exercise 4.1.13, and let x be its maximal point. Show that in some but not all cases $k\widetilde{\to}\mathcal{O}_{X,x}$. Thus, the usual topology is too crude to allow non-constant functions in a neighborhood of x.

A simplest separated example of a non-good space is as follows.

Example 4.2.4. Assume that $\underline{r}$ is a tuple of $n > 1$ positive numbers. A closed polydisc $E(0, \underline{r})$ of polyradius $\underline{r}$ with removed open polydisc of polyradius $\underline{r}$ is a compact not good analytic domain in $E(0, \underline{r})$.

4.2.2. *Finite morphisms, closed immersions and Zariski topology.*

Definition 4.2.5. A morphism $f : Y \to X$ is called a *closed immersion* (resp. *finite*) if there exists an admissible covering by affinoid domains $X = \cup_i X_i$ such that $Y_i = X_i \times_X Y$ are k-affinoid and the homomorphism of Banach algebras $\mathcal{O}_{X_G}(X_i) \to \mathcal{O}_{Y_G}(Y_i)$ is finite (resp. surjective) and admissible.

We have already discussed why admissibility is essential when the valuation on k is trivial.

Exercise 4.2.6. (i) For any affinoid domain $X' \subseteq X$ the base change morphism $X' \times_X Y \to X'$ is a closed immersion (resp. finite). (Hint: $\mathcal{O}_{Y_G}$ is a coherent $\mathcal{O}_{X_G}$-algebra.)

(ii) The class of closed immersions (resp. finite morphisms) is closed under compositions, base changes and ground field extensions.

Definition 4.2.7. Any subset $Z \subseteq X$ that is the image of a closed immersion is called *Zariski closed*. The complement of such set is called *Zariski open*.

Exercise 4.2.8. A point $x \in X$ is Zariski closed if and only if $[\mathcal{H}(x) : k] < \infty$. (Zariski closed points of X are precisely its classical rigid points. The set of all such points is denoted X_0.)

When working with Zariski topology one must be very careful because it becomes stronger when passing to analytic domains (even open ones). In other words, coherent ideals on an open subspace do not have to extend to the whole space. Such phenomenon does not occur for algebraic varieties (for obvious reasons) but does occur for formal varieties.

Example 4.2.9. (i) Give an example of a k-analytic space X with an open subspace U and a closed subspace $Z \subset U$ which does not extend to the whole X. (Hint: take $X = \mathcal{M}(k\{T, S\})$ the unit polydisc, U a closed polydisc of polyradius $(1, r)$ with $r < 1$ and Z given by $T - f(S)$, where $f(S)$ has radius of convergence between r and 1.)

(ii) An example of Ducros. Fix $r \notin \sqrt{|k^\times|}$ with $0 < r < 1$. Consider a polydisc $X = \mathcal{M}(k\{T, S\})$ with an affinoid domain $V = \mathcal{M}(k\{r^{-1}T, rT^{-1}, S\})$, which is a unit K_r-disc and a non-strict k-surface (the product of the unit k-disc with the irrational k-annulus $\mathcal{M}(K_r)$). Using the hint from (i) find a Zariski closed point $x \in V$ with $\mathcal{H}(x) \widetilde{\to} K_r$ such that the ideal of x in V does not extend to any neighborhood of x in X. (In a sense, x is a k-curve in the k-surface X which cannot be extended, so x is Zariski closed only in a sufficiently small domain in X.) Show that in this case $\mathcal{O}_{X,x}$ is a dense subfield of K_r and hence the character $\chi_{X,x} : k\{T, S\} \to \mathcal{H}(x)$ is injective and hence flat. On the other hand, its base change with respect to the homomorphism $k\{T, S\} \to k\{r^{-1}T, rT^{-1}, S\}$ is not flat because the character $\chi_{V,x}$ has a non-trivial kernel. In particular, one cannot define a reasonable class of flat morphisms between k-affinoid spaces just by saying that $\mathcal{M}(f)$ is flat whenever f is flat. However, one can show that this approach works well for strictly k-affinoid spaces.

4.2.3. *Separated morphisms.*

Definition 4.2.10. (i) A morphism $f : Y \to X$ is *separated* if the diagonal morphism $\Delta_{Y/X} : Y \to Y \times_X Y$ is a closed immersion.

(ii) A k-analytic space is *separated* if so is its morphism to $\mathcal{M}(k)$.

Exercise 4.2.11. (i) Formulate and prove the basic properties of separated morphisms analogous to the properties of separated morphisms of schemes. In particular, show that in a separated k-analytic space X the intersection of two k-affinoid domains is k-affinoid.

(ii) Prove Fact 4.1.10 for the subcategories of separated objects in k_H-*An* and k-*An*.

Non-separatedness of a space can be of two sorts, as is illustrated by the following example.

Example/Exercise 4.2.12. (i) Let X be the closed unit disc with doubled open unit disc. Show that X is not locally separated at its maximal point x (i.e. any k-analytic domain which is a neighborhood of x is not separated). In particular, X is a non-good k-analytic Hausdorff space.

(ii) Show that the closed unit disc Y with doubled origin is not separated but is locally separated at all its points. Moreover, Y is a good non-Hausdorff k-analytic space.

4.2.4. *Boundary and proper morphisms.*

Definition 4.2.13. (i) The *relative interior* $\mathrm{Int}(Y/X)$ of a morphism $f : Y \to X$ is the set of all points $y \in Y$ such that any for affinoid domain $U \subseteq X$ containing $x = f(y)$ there exists an affinoid domain $V \subseteq f^{-1}(x)$ such that V is a neighborhood of x (in particular, $f^{-1}(U)$ is good) and $x \in \mathrm{Int}(Y/X)$. The complement $\partial(Y/X) = Y \setminus \mathrm{Int}(Y/X)$ is called the *relative boundary* and we say that f is *boundaryless* or without boundary if $\partial(Y/X)$ is empty (in [Ber2] such morphisms are called "closed").

(ii) A morphism $f : Y \to X$ is *proper* if it is boundaryless and compact (i.e. the preimage of a compact domain is compact).

As usual, the absolute analogs of these notions are defined relatively to $\mathcal{M}(k)$. Note that (part (i) of) this definition agrees with our earlier definitions from §3.4.2.

Example/Exercise 4.2.14. (i) A k-analytic space has no boundary if and only if any its point x possesses an affinoid neighborhood U such that $x \in \mathrm{Int}(U)$. For example, an open polydisc, $\mathbf{P}^n_k$ and $\mathbf{A}^n_k$ have no boundary, and a closed polydisc has a boundary. So far, $\mathbf{P}^n_k$ is the only example of a proper non-discrete k-analytic space we have considered. It follows that any *projective k-analytic space*, i.e. a closed subspace of $\mathbf{P}^n_k$ is also proper.

(ii) Any boundaryless k-analytic space X is good. If the valuation on k is non-trivial then X is also strict.

(iii) A morphism between affinoid spaces is proper if and only if it is finite. Any finite morphism is proper.

(iv) A boundaryless morphism is separated if and only if the preimage of any Hausdorff domain is Hausdorff. In particular, proper morphisms are separated.

Fact 4.2.15. (i) If Y is an analytic domain in X then $\mathrm{Int}(Y/X)$ is the topological interior of Y in X.

(ii) Boundaries are G-local on the base, i.e. given a locally finite covering of X by affinoid domains X_i one has that $\partial(Y/X) = \cup_i \partial(X_i \times_X Y/X_i)$.

(iii) The classes of proper morphisms and morphisms without boundary are G-local on the base and are preserved by under compositions, base changes and ground field extensions.

(iv) If $f : Y \to X$ is a separated boundaryless morphism and X is k-affinoid then for any affinoid domain $U \subseteq Y$ there exists a larger k-affinoid domain $V \subseteq Y$ such that $U \subseteq \mathrm{Int}(V/X)$ and U is a Weierstrass domain in V.

Remark 4.2.16. Surprisingly enough, already (ii) and the result about compositions are really difficult. It turns out that when one wants to show that various morphisms have no boundary, the difficult part of the proof is to show that the preimage of an affinoid domain under these morphisms is a good domain. Since this is established, one can use the theory of boundaries for affinoid spaces as outlined in §3.4.2. See also Remark 4.2.18 below.

Similarly to algebraic and complex analytic geometries, coherence is preserved by higher direct images with respect to proper morphisms.

Fact 4.2.17 (Kiehl's theorem on direct images)**.** If $f : Y \to X$ is a proper morphism between k-analytic spaces and $\mathcal{F}$ is a coherent $\mathcal{O}_{Y_G}$-modules then the $\mathcal{O}_{X_G}$-modules $R^i f_*(\mathcal{F})$ are coherent.

Note that we use here that f is a compact map because otherwise $f_*(\mathcal{F})$ is not a sheaf of Banach $\mathcal{O}_{X_G}$-modules.

Remark 4.2.18. Kiehl introduced the notion of proper morphisms and proved the above result (for rigid spaces) in [Ki]. One can easily show that our definition of proper morphisms (in the strict case) is equivalent to the original Kiehl's definition. The definition of proper morphism is designed so that the theorem on direct images can be proved rather easily and naturally (one computes Čech complexes and shows that certain differentials are compact operators). As was already remarked, it is very difficult to establish some other properties, that one might expect to be more foundational. For example, the fact that proper morphisms are preserved by compositions was open for more than twenty years (for a discrete valued k this was proved in [Lüt] and the general case was established in [Tem1] and [Tem2]).

4.2.5. *Smooth and étale morphisms.*

Definition 4.2.19. (i) A finite morphism $f : Y \to X$ is *étale* if for any affinoid domain $U \subseteq X$ and its preimage $V = f^{-1}(U)$ the finite homomorphism of k-affinoid algebras $\mathcal{O}_{X_G}(U) \to \mathcal{O}_{Y_G}(V)$ is étale. (We know that V is affinoid and finite over U.)

(ii) In general, a morphism $f : Y \to X$ is *étale* if locally (on Y) it is finite étale. Namely, for any point $y \in Y$ there exist neighborhoods V of y and U of $f(y)$ such that f restricts to a finite étale morphism $V \to U$.

(iii) A morphism $f : Y \to X$ is *smooth* if it can be represented as an étale morphism $Y \to \mathbf{A}^n_X$ followed by the projection.

Exercise 4.2.20. (i) Any smooth morphism (e.g. an étale morphism) is boundaryless.

(ii) If V is an analytic domain in X then the embedding $V \hookrightarrow X$ is étale if and only if it is an open immersion.

(iii)* Any smooth morphism is an open map.

Fact 4.2.21. The classes of étale and smooth morphisms are closed under compositions, base changes and ground field extensions. Also these classes are G-local on the base (the étale case follows from Kiehl's theorem; Fact 4.2.15(ii) is the main ingredient in the proof of the smooth case).

This definition of étale and smooth morphisms is analogous to a complex analytic definition but it does not apply to nice morphisms with boundaries. For example, the closed unit disc is not smooth at its maximal point. The following definition is a natural generalization to the case when there are boundaries. We give it for the sake of completeness, but do not discuss all results one should prove to show that it really makes sense.

Definition 4.2.22. (i) A morphism $f : Y \to X$ between strictly k-analytic spaces is *rig-smooth* if the restriction of f on $\mathrm{Int}(Y/X)$ is smooth.

(ii) In general, a morphism $f : Y \to X$ is *rig-smooth* if so is some (and then any) ground field extension $f_{\underline{r}} := f\widehat{\otimes}_k K_{\underline{r}}$ such that $Y_{\underline{r}}$ and $X_{\underline{r}}$ are strictly K_r-analytic.

(iii) A rig-smooth morphism with discrete fibers is called *quasi-étale*.

Remark 4.2.23. (i) Alternatively, one can define quasi-étale morphisms directly and then rig-smooth morphisms are the morphisms that locally split into the composition of a quasi-étale morphism with the projection $\mathbf{A}^n_X \to X$.

(ii) We have to extend the ground field in the general case because $\mathrm{Int}(Y/X)$ can be too small to test (any sort of) smoothness. A good example of such situation was studied in Exercise 4.2.9(ii).

(iii) The same problem happens when one wants to introduce flatness. A reasonable theory of flatness was developed very recently by Ducros. In the strict case one gives a naive definition, and in general f is called *flat* if so is its strictly analytic ground field extension.

(iv) I expect that f is rig-smooth if and only if it is flat (in Ducros' sense) and the coherent $\mathcal{O}_{Y_G}$-module of continuous differentials $\Omega^1_{Y_G/X_G}$ is locally free (this sheaf admits the following local description: if $X = \mathcal{M}(\mathcal{A})$ and $Y = \mathcal{M}(\mathcal{B})$ then $\Omega^1_{Y_G/X_G}$ corresponds to the module I/I^2 where $I = \mathrm{Ker}(\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{B} \to \mathcal{B})$).

4.3. **Basic topological properties.**

4.3.1. *A general conjecture.* We noticed in §2.3.3 that the affine line is a sort of an infinite tree. The topological structure of general analytic spaces is still unclear and it is a subject of intensive research activity. We will construct in Exercise 6.1.10 an example of a k-analytic curve C such that C is a double covering of an open unit disc and C is a closed subspace in a two-dimensional open unit polydisc but the first Betti number of C is infinite (actually, C is a sort of an infinite graph with infinitely many loops). This forces us to consider only compactifiable spaces in the following conjecture.

Conjecture 4.3.1. *Let X be an n-dimensional connected* compactifiable *k-analytic space. (The latter condition means that X is isomorphic to an analytic domain in a compact analytic space $\overline{X}$.) Then X contains a family of topological spaces $\{S_i\}_{i\in I}$ filtered by inclusion such that each S_i is homeomorphic to a finite simplicial complex of dimension at most n and there exists a projective family of maps $f_{ij} : S_i \to S_j$ (for each pair $S_j \subseteq S_i$) such that $X \widetilde{\to} \mathrm{proj\,lim}_{i\in I} S_i$. Moreover, this family extends to a compatible family of deformational retractions $\Phi_{ij} : S_i \times [0,1] \to S_j$ with $\Phi_{ij}(x,1) = f_{ij}(x)$ which induce deformational retractions of X on each of S_i's.*

Recently, Hrushovski and Loeser proved this for any X which is an analytic domain in (the analytification of) an algebraic k-variety (no paper is available yet). Below we only discuss published facts about the topology of analytic spaces.

4.3.2. *Pathwise connectedness and dimension.*

Fact 4.3.2. (i) Any connected k-analytic space is pathwise connected.

(ii) Any point has a fundamental family of neighborhoods which are compact and pathwise connected analytic domains.

(iii) The topological dimension of X is at most $\dim(X)$ and both are equal in the strict case.

This fact was proved in [Ber1] (and the argument is correct, although by a misunderstanding some mathematicians thought that part (i) was not proved). Let us say few words about the proof of (i). We have already checked this fact for an affine line and one easily deduces the case of a polydisc. Studying finite covers of polydiscs one obtains the case of strictly k-analytic spaces. The general case is deduced by descent from an appropriate $X\widehat{\otimes}_k K_{\underline{r}}$.

4.3.3. *Contractions.* In some cases one can construct by hands a retraction of a k-analytic space X onto a subset S which is of topologically finite type. One such method is to find an action of a k-affinoid group G on X with a continuous family of affinoid subgroups $\{G_t\}_{t\in[0,1]}$ such that $G_0=\{e\}$, $G_1=G$ and for each point $x\in X$ each orbit G_tx is affinoid and possesses exactly one maximal point x_t. Then $(x,t)\mapsto x_t$ defines a retraction of X onto some its subset, which is very small in some examples. Two good examples of such G are as follows a closed unit polydisc $\mathcal{M}(k\{\underline{T}\})$ with an additive group structure and a product of unit annuli $G^n_{m,1}=\mathcal{M}(k\{T_1,T_1^{-1},\dots,T_n,T_n^{-1}\})$ with the multiplicative group structure. The groups G_t with $t<1$ are the polydiscs of polyradius $(t,\dots,t)$ with center at 0 or 1, respectively. The action of the torus is much more important because tori play important role in the theory of reductive groups (see [Ber1, §5] for the connection to Bruhat-Tits buildings). So, we give the most fundamental example of a contraction by a torus action.

Example/Exercise 4.3.3. (i) Show that $\mathbf{R}^n_+$ embeds into $\mathbf{A}^n_k$ so that $\underline{r}=(r_1,\dots,r_n)$ goes to the semivaluation $\|\ \|_{\underline{r}}$ (the maximal point of $E(0,\underline{r})$).

(ii)* Show that the action of $G^n_{m,1}$ on $\mathbf{A}^n_k$ contracts it onto $\mathbf{R}^n_+$. Moreover, the retraction can be explicitly described by the formula $|f(x_t)|=\max_{i\in\mathbf{N}^n}|\partial_i f(x)|t^i$ where $f(\underline{T})\in k[T_1,\dots,T_n]$ and $\partial_i:k[\underline{T}]\to k[\underline{T}]$ for $i\in\mathbf{N}^n$ is the logarithmic differential operator $\frac{\underline{T}^i}{i!}\frac{\partial^i}{d\underline{T}^i}$.

Finally, we mention the following very difficult result of Berkovich whose proof is a subject of a separate paper.

Fact 4.3.4. Any analytic domain in a smooth k-analytic space is locally contractible.

5. Relation to other categories

5.1. Analytification of algebraic k-varieties.

5.1.1. *The analytification functor.* Let k-Var be the category of algebraic k-varieties (i.e. schemes of finite type over k). We are going to describe a construction of an analytification functor k-$Var\to k$-An. The analytification of a morphism $f:\mathcal{Y}\to\mathcal{X}$ will be denoted $f^{\mathrm{an}}:\mathcal{Y}^{\mathrm{an}}\to\mathcal{X}^{\mathrm{an}}$. For $\mathcal{X}=\mathrm{Spec}(k[T_1,\dots,T_n])$ we set $\mathcal{X}^{\mathrm{an}}=\mathbf{A}^n_k=\mathcal{M}(k[\underline{T}])$. For any quotient $A=k[\underline{T}]/I$ the analytification of $\mathcal{Y}=\mathrm{Spec}(A)$ is the closed subspace of $\mathcal{X}^{\mathrm{an}}$ defined by vanishing of $I\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}$.

Exercise 5.1.1. (i) Prove that this definition is independent of choices. Also, show that $\mathcal{Y}^{\mathrm{an}}=\mathcal{M}(A)$ is the set of all real semivaluations on A bounded on k. (Hint: two embeddings of $\mathcal{Y}$ into affine spaces are dominated by a third such embedding.)

(ii) Extend this to a functor from the category of affine k-varieties to the category of boundaryless k-analytic spaces.

(iii) Show that the latter functor takes open immersions to open immersions and hence extends (by gluing) to an analytification functor $k\text{-}Var \to k\text{-}An$. Show that any analytification is a boundaryless space.

(iv) Show that $(\mathrm{Proj}(A))^{\mathrm{an}} \widetilde{\to} \mathbf{P}\mathcal{M}(A)$. In particular, $\mathbf{P}\mathcal{M}(A)$ is a projective analytic variety.

Fact 5.1.2. The analytification functor can be described via the following universal property. For any good k-analytic space Y let $F_{\mathcal{X}}(Y)$ be the set of morphisms of locally ringed spaces $(Y, \mathcal{O}_Y) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Then $X = \mathcal{X}^{\mathrm{an}}$ is the k-analytic space that represents $F_{\mathcal{X}}$.

In particular, a morphism $\pi\mathcal{X} : (\mathcal{X}^{\mathrm{an}}, \mathcal{O}_{\mathcal{X}^{\mathrm{an}}} \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ arises.

Fact 5.1.3. $\mathcal{X}^{\mathrm{an}}(K) \widetilde{\to} \mathcal{X}(K)$ for any non-archimedean k-field K, in particular $\pi\mathcal{X}$ is surjective.

Definition 5.1.4. For a coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ the module $\mathcal{F}^{\mathrm{an}} = \pi_{\mathcal{X}}^*(\mathcal{F})$ is a coherent $\mathcal{O}_{\mathcal{X}^{\mathrm{an}}}$-module called the *analytification* of $\mathcal{F}$.

5.1.2. *GAGA.* The analytification functor preserves almost all properties of varieties and their morphisms and here is a (partial) list.

Fact 5.1.5. Let $f : \mathcal{Y} \to \mathcal{X}$ be a morphism between algebraic k-varieties. Then f satisfies one of the following properties if and only if so does f^{an}: smooth, étale, finite, closed immersion, open immersion, isomorphism, proper, separated.

Fact 5.1.6. For a proper variety $\mathcal{X}$ the analytification functor induces an equivalence $\mathrm{Coh}(\mathcal{O}_{\mathcal{X}}) \widetilde{\to} \mathrm{Coh}(\mathcal{O}_{\mathcal{X}^{\mathrm{an}}})$.

Exercise 5.1.7. (i) The functor $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{an}}$ is fully faithful on the category of proper varieties, but not in general.

(ii) For a proper variety $\mathcal{X}$, the analytification functor induces an equivalence between the categories of finite (resp. finite étale) $\mathcal{X}$-scheme and $\mathcal{X}^{\mathrm{an}}$-spaces.

(iii) Any projective k-analytic space X is algebraizable by a projective k-variety $\mathcal{X}$ (i.e. $X \widetilde{\to} \mathcal{X}^{\mathrm{an}}$).

When the valuation on k is trivial, the properness assumption can be eliminated. Let $X_t \subset X$ be the set of points $x \in X$ with trivially valued completed residue field $\mathcal{H}(x)$.

Exercise 5.1.8. Assume that the valuation on k is trivial.

(i) $\mathcal{X}_t^{\mathrm{an}} \widetilde{\to} \mathcal{X}$.

(ii) The analytification functor is fully faithful.

(iii) For a variety $\mathcal{X}$ the analytification functor induces an equivalence of categories $\mathrm{Coh}(\mathcal{O}_{\mathcal{X}}) \widetilde{\to} \mathrm{Coh}(\mathcal{O}_{\mathcal{X}^{\mathrm{an}}})$.

5.2. **Generic fibers of formal k°-schemes.**

5.2.1. *Reminds on formal schemes.*

Definition 5.2.1. Let A be a ring with an ideal I.

(i) The *I-adic topology* on A is generated by the cosets $a + I^n$.

(ii) The *separated I-adic completion* is defined as $\widehat{A} = \mathrm{proj}\lim_n A/I^n$.

(iii) A is *I-adic* if $A\widetilde{\to}\widehat{A}$.

(iv) Any ideal J with $I \subseteq J^n$ and $J^n \subseteq I$ for large enough n is called *ideal of definition* of A. (It generates the same topology and can be used instead of I in all definitions.)

Example/Exercise 5.2.2. (i) If k° is a real valuation ring with fraction field k and $\pi \in k^{\circ\circ}$ is any non-zero element then the (π)-adic completion of k° is the ring of integers $\widehat{k}^\circ$ of the completion of k.

(ii) The separated $(k^{\circ\circ})$-completion of k is either $\widehat{k}^\circ$ or $\widetilde{k}$. Moreover, the first possibility occurs only when k is discrete or trivially valued.

Definition 5.2.3. (i) The *formal spectrum* $\mathfrak{X} = \mathrm{Spf}(A)$ of an I-adic ring A is the set of closed ideal of A with the topology generated by the sets $D(f)$, where $D(f)$ is the non-vanishing locus of an element $f \in A$.

(ii) Each $D(f)$ is homeomorphic to $\mathrm{Spf}(A_{\{f\}})$, where the *formal localization* $A_{\{f\}}$ is the universal I-adic A-algebra with inverted f.

(iii) The structure sheaf $\mathcal{O}_\mathfrak{X}$ is a sheaf of topological rings determined by the condition $\mathcal{O}_\mathfrak{X}(D(f)) = A_{\{f\}}$. The topologically ringed space $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ is called the *affine formal scheme* associated with A.

(iv) The *closed fiber* (or special fiber) $\mathfrak{X}_s$ of $\mathfrak{X}$ is the reduction of $\mathrm{Spec}(A/J)$ for any ideal of definition J.

Exercise 5.2.4. (i) Show that $A_{\{f\}}\widetilde{\to}\operatorname{proj\,lim}_n(A/I^n)_f$ and $A\{T\}/(Tf-1)\widetilde{\to}A_{\{f\}}$, where $A\{T\} = \operatorname{proj\,lim}_n(A/I^n)[T]$ is the ring of convergent power series over A.

(ii) If $\pi^n \in I$ for some n then $A_{\{\pi\}} = 0$. (Thus formal localization at topologically nilpotent element has the same effect as inverting a nilpotent element in a ring.)

(iii) Show that $\mathfrak{X}_s$ does not depend on the ideal of definition and $|\mathfrak{X}_s|\widetilde{\to}|\mathfrak{X}|$. Actually, $\mathfrak{X}$ can be viewed as the inductive limit of schemes $(\mathfrak{X}_s, \mathcal{O}_\mathfrak{X}/J)$ where J runs through the ideals of definition.

Definition 5.2.5. A general *formal scheme* is a topologically ringed space $(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$ which is locally isomorphic to affine formal schemes. Morphisms of such creatures are certain morphisms of topologically ringed spaces (the restriction is that the homomorphisms on the ring-theoretical stalks are local homomorphisms.) Ideals of definitions and the closed fiber are

Definition/Exercise 5.2.6. (i) The n-dimensional affine space over an adic ring A is defined as $\mathbf{A}^n_A = \mathrm{Spf}(A\{T_1, \dots, T_n\})$.

(ii) A formal scheme over an I-adic ring A is of *(topologically) finite presentation* (resp. *special*) if it is locally of the form $\mathrm{Spf}(A\{T_1, \dots, T_n\}/(f_1, \dots, f_m))$ (resp. $\mathrm{Spf}(A[T_1, \dots, T_n][[S_1, \dots, S_l]]/(f_1, \dots, f_m)))$ and the topology on A is I-adic (resp. $(I, S_1, \dots, S_l)$-adic).

5.2.2. *Generic fibers of formal k°-schemes of finite type.* In this section we are going to define a *generic fiber functor* η which assigns to a formal k°-scheme $\mathfrak{X}$ of locally finite type a Hausdorff strictly k-analytic space $\mathfrak{X}_\eta$ (even when the valuation is trivial). Intuitively, $\mathfrak{X}_\eta$ is the "missing generic fiber of $\mathfrak{X}$" and when k is non-trivially valued it is defined by inverting a non-zero element $\pi \in k^{\circ\circ}$. (By Exercise 5.2.4(ii) we kill any formal k°-scheme by such an operation, so it is not surprising that $\mathfrak{X}_\eta$ is not a formal scheme but leaves in another category.) If k is trivially valued then we set $\pi = 0$ to uniformize the exposition.

In general, the definition of η is very similar to the definition of the analytification. One defines $(\mathbf{A}^n_{k^\circ})_\eta$ to be the closed unit polydisc $E^n(0,1)$. An affine scheme given by vanishing of $f_1,\dots,f_n$ in $\mathbf{A}^n_{k^\circ}$ is defined as the closed subspace in $E^n(1,0)$ given by vanishing of f_i's. In general, η is defined via gluing. Let us realize this program with some details.

Definition 5.2.7. If $A = k^\circ\{T_1,\dots,T_n\}/I$ is a π-adic ring with a finitely generated I then $\mathcal{A} = A_\pi = A\otimes_{k^\circ} k$ is a k-affinoid algebra isomorphic to $k\{\underline{T}\}/Ik\{\underline{T}\}$. For the affine formal scheme $\mathfrak{X} = \mathrm{Spf}(A)$ we set $\mathfrak{X}_\eta = \mathcal{M}(\mathcal{A})$.

Exercise 5.2.8. Assume that A has no π-torsion, and so A embeds into $\mathcal{A}$.

(i) The integral closure of A in $\mathcal{A}$ is $\mathcal{A}^\circ$.

(ii) If the valuation on k is non-trivial and A is reduced then A is the unit ball for a Banach norm $|\ |$ on $\mathcal{A}$. This means that $|\ |$ is equivalent to $\rho_\mathcal{A}$ or, equivalently, $\pi^n\mathcal{A}^\circ\subseteq A\subseteq\mathcal{A}^\circ$ for large enough n.

(iii) Formal localization is compatible with inverting π. Namely, $(A_{\{f\}})\pi\widetilde{\to}A_\pi\{f^{-1}\}$. (Hint: use Fact 3.1.5.)

The exercise (iii) above implies that η (defined for affine formal schemes) takes open immersions to embeddings of affinoid domains. Now we can define the functor η in general.

Definition/Exercise 5.2.9. (i) If a separated formal scheme $\mathfrak{X}$ of locally finite type over k° is glued from open subschemes $\mathfrak{X}_i$ along the intersections $\mathfrak{X}_{ij}$ then the gluing of $(\mathfrak{X}_i)\eta$ along $(\mathfrak{X}_{ij})_\eta$ is possible by Exercise 4.1.13(ii) (we use the modified definition of local finite type morphisms) and the obtained k-analytic space is defined to be $\mathfrak{X}_\eta$.

(ii) For a general formal scheme $\mathfrak{X}$ we repeat the same construction but with X_{ij} being separated formal schemes now.

(iii) Check that this construction defines the promised generic fiber functor (in particular, it extends to morphisms).

As one might expect, η preserves (or naturally modifies) almost all properties of morphisms.

Fact 5.2.10. Let $f:\mathfrak{Y}\to\mathfrak{X}$ be a morphism between formal k°-schemes without π-torsion. If f is an isomorphism, separated, proper, a closed immersion, finite étale, then f_η is so. If f is an open immersion, étale, or smooth, then f_η is a compact embedding of analytic domain, quasi-étale, or rig-smooth, respectively.

The claim about properness is really difficult (the definition of properness in k-An is such that it is even difficult to see that properness is preserved by compositions). Intuitively, one cannot expect that something can be proved in the opposite direction (e.g. a generically finite morphism does not have to be finite). However, the following result holds true.

Fact 5.2.11. If f_η is proper or separated then so is f.

Finally, there exists an anti-continuous *reduction map* $\pi_\mathfrak{X}:\mathfrak{X}_\eta\to\mathfrak{X}_s$ defined similarly to the affinoid reduction map.

Definition/Exercise 5.2.12. Check that for any point $x\in\mathfrak{X}_\eta$ its character $\chi_x:\mathcal{M}(\mathcal{H}(x))\to\mathfrak{X}_\eta$ is induced by a morphism $\chi^\circ_x:\mathrm{Spf}(\mathcal{H}(x)^\circ)\to\mathfrak{X}$. This induces a point $\widetilde{\chi}_x:\mathrm{Spec}(\widetilde{\mathcal{H}(x)})\to\mathfrak{X}_s$ on the closed fiber and hence gives rise to a map $\pi_\mathfrak{X}$.

Remark 5.2.13. The reduction map is an analog of the following specialization construction. If X is a scheme over a henselian valuation ring (e.g. k°), $(X_\eta)_0$ is the set of closed points of the generic fiber and $(X_s)_0$ is the set of closed points of the closed fiber then specialization induces a map $(X_\eta)_0 \to (X_s)_0$.

5.2.3. *Relation to the analytification.* If k is trivially valued then analytification and generic fiber provides two functors from the category of k-varieties to the category of k-analytic spaces. More generally, for any k we have two functors $\mathcal{F}$ and $\mathcal{G}$ from the category of k°-schemes of finite type to the category of k-analytic spaces: $\mathcal{G}(X) = (X_\eta)^{\mathrm{an}}$ and $\mathcal{F}(X) = (\widehat{X})_\eta$. In the first case, we first pass to the generic fiber of the morphism $X \to \mathrm{Spec}(k^\circ)$ and then analytify the obtained k-variety. In the second case, we first complete X and then take the generic fiber of the obtained formal k°-scheme of finite type.

Exercise 5.2.14. (i) Assume that $X = \mathrm{Spec}(A)$ and $f_1, \dots, f_n$ generate A over k°. Show that $(\widehat{X})_\eta$ can be naturally identified with the affinoid domain in (the Stein space) $(X_\eta)^{\mathrm{an}}$ defined by the conditions $|f_i| \le 1$.

(ii) Extend this construction to a morphism of functors $\phi : \mathcal{F} \to \mathcal{G}$ which is a compact embedding of a strictly analytic domain.

(iii) Show that when restricted to proper k°-schemes ϕ induces an isomorphism of functors, i.e. the embedding of the analytic domain $\phi(X) : (\widehat{X})_\eta \hookrightarrow (X_\eta)^{\mathrm{an}}$ is an isomorphism for a k°-proper X.

5.2.4. *Generic fibers of k°-special formal schemes.* For completeness, we discuss briefly how the generic fiber functor extends to all k°-special formal schemes in the case of a discrete valued (or trivially valued) field k. (The non-discrete valued case was not studied in the literature because the rings $k^\circ[[T_1, \dots, T_n]])$ are rather pathological then, e.g. they possess non-closed ideals.)

The general idea of defining $\mathfrak{X}_\eta$ is actually the same: for

$$\mathfrak{X} = \mathrm{Spf}(k^\circ[T_1, \dots, T_n][[T_{n+1}, \dots, T_m]])$$

one defines $\mathfrak{X}_\eta \subseteq \mathcal{M}(k\{T_1, \dots, T_m\})$ as the unit polydisc given by the conditions $|T_i| < 1$ for $n < i \le m$. In particular, the polydisc is open when $n = 0$ and is closed when $n = m$. For an affine $\mathfrak{Y}$ one defines $\mathfrak{Y}_\eta$ using a closed embedding into $\mathfrak{X}$ as above, and for a general special formal scheme the functor is defined using gluing. An anti-continuous reduction map $\mathfrak{X}_\eta \to \mathfrak{X}_s$ is defined as earlier.

Exercise 5.2.15. (i) $\mathfrak{X}_\eta$ is a good k-analytic space, and it is strict when the valuation is non-trivial.

(ii) In the affine case $\mathfrak{X} = \mathrm{Spf}(A)$ the generic fiber can be identified with the set of real semivaluations on A that extend the valuation on A, are bounded and are strictly smaller than one on the element of an ideal of definition of A.

(iii) In the affine case, $\mathfrak{X}_\eta$ is an increasing union of affinoid domains X_n such that X_n is Weierstrass in each X_m for $m \ge n$ (e.g. an open polydisc is an increasing union of smaller closed polydiscs). (In particular, $\mathfrak{X}_\eta$ is a Stein space.)

One of motivations to introduce generic fibers of special fibers is the following result of Berkovich.

Fact 5.2.16. Let $\mathfrak{X}$ be a k°-special formal scheme (e.g. a formal scheme of finite type over k°) and let $Z \hookrightarrow \mathfrak{X}_s$ be a closed subscheme. Then the preimage of Z under the reduction map $\pi_{\mathfrak{X}} : \mathfrak{X}_\eta \to \mathfrak{X}_s$ depends only on the formal completion of $\mathfrak{X}$ along $\mathfrak{Z}$. Moreover, this preimage is precisely the generic fiber $(\widehat{\mathfrak{X}}_Z)_\eta$ of the formal completion.

The following result in the opposite direction was proved in [Tem5] (and applied to resolution of singularities in positive characteristic). It describes the precise information about the formal scheme that is kept in the generic fiber $(\widehat{\mathfrak{X}}_Z)_\eta$.

Fact 5.2.17. If $\mathfrak{X}$ is locally of the form $\mathrm{Spf}(\mathcal{A}^\circ)$ for a k-affinoid algebra $\mathcal{A}$ (i.e. $\mathfrak{X}$ is of finite type over k° and is normal in its generic fiber), then the henselization of $\mathfrak{X}$ along a closed subscheme $Z \hookrightarrow \mathfrak{X}_s$ is completely determined by the generic fiber $(\widehat{\mathfrak{X}}_Z)_\eta$.

5.3. **Raynaud's theory.**

5.3.1. *An overview.* Assume that the valuation is non-trivial. We constructed a functor η whose source is the category k°-$Fsch$ of formal k°-schemes of finite type and whose target is the category *st*-k-An^c of compact strictly k-analytic spaces. Raynaud's theory completely describes this functor in the following terms: η is a localization of the target by an explicitly given family of morphisms $\mathcal{B}$. In particular, one can view a compact strictly k-analytic space$\mathfrak{X}_\eta$ as its *formal model* $\mathfrak{X}$ given up to a morphism from $\mathcal{B}$. (A possible analogy is to think about $\mathfrak{X}$ as a particular atlas of a manifold $\mathfrak{X}_\eta$ with morphisms from $\mathcal{B}$ being the refinements of the atlases.)

Clearly, the central part of the theory should be to describe the morphisms $f : \mathfrak{Y} \to \mathfrak{X}$ that are rig-isomorphisms (or generic isomorphisms). Although this is not so easy, we will find a nice cofinal family $\mathcal{B}$ among all rig-isomorphisms. To guess what such $\mathcal{B}$ can be, let us consider a very similar problem in the theory of schemes. Given a scheme X with a schematically dense open subscheme U (which will play the role of the generic fiber) by a U-modification of X we mean a proper morphism $f : X' \to X$ such that $f^{-1}(U)$ is schematically dense in X' and is mapped isomorphically onto U. A strong version of Chow lemma states that the family of U-admissible blow ups, i.e. blow ups long center disjoint from U, form a cofinal family among all U-modifications. Now, it is natural to expect that one can take $\mathcal{B}$ to be the family of all formal blow ups along open ideals (i.e. ideals supported on $\mathfrak{X}_s$).

5.3.2. *Admissible blow ups.*

Definition 5.3.1. Recall that the blow up $\mathrm{Bl}_\mathcal{I}(X)$ of a scheme X along an ideal $\mathcal{I} \subseteq \mathcal{O}_X$ is defined as $\mathbf{Proj}(\oplus_{n=0}^\infty)\mathcal{O}_X/\mathcal{I}^n$. If A is an I-adic ring then the formal blow up of the affine formal scheme $\mathrm{Spf}(A)$ along an ideal $J \subseteq A$ is defined as the I-adic completion of $\mathrm{Bl}_J(\mathrm{Spec}(A))$. This definition is local on the base and hence globalizes to a definition of *formal blow up* $\widehat{\mathrm{Bl}}_\mathcal{J}(\mathfrak{X})$ of a formal scheme $\mathfrak{X}$ along an ideal $\mathcal{J} \subseteq \mathcal{O}_\mathfrak{X}$. If $\mathcal{J}$ is open (i.e. contains an ideal of definition) then the formal blow up is called *admissible.*

Fact 5.3.2. (i) Any composition of (admissible) formal blow ups is an (admissible) formal blow up.

(ii) (Admissible) blow ups form a filtered family.

Exercise 5.3.3. If $\mathfrak{X}$ is of finite type over k° then any admissible formal blow up is a rig-isomorphism.

5.3.3. *The main results.*

Fact 5.3.4 (Raynaud)**.** The family $\mathcal{B}$ of formal blow ups in the category k°-$Fsch$ admits a calculus of right fractions and the localized category is equivalent to *st*-*k*-An^c. The localization functor is isomorphic to the generic fiber functor.

Remark 5.3.5. This fact implies the following two corollaries:

(i) The family of admissible blow ups of a formal scheme $\mathfrak{X}$ from k°-$\mathcal{F}sch$ is cofinal in the family of all rig-isomorphisms $\mathfrak{X}' \to \mathfrak{X}$.

(ii) Each compact strictly k-analytic space X admits a *formal model* $\mathfrak{X}$, i.e. a formal scheme $\mathfrak{X}$ in k°-$\mathcal{F}sch$ with an isomorphism $\mathfrak{X}_\eta \to X$.

Actually, these two statements serve as intermediate steps while proving Fact 5.3.4. Moreover, one proves that if $\{X_i\}$ is finite family of compact strictly analytic domains in X then one can find a model $\mathfrak{X}$ with open subschemes $\mathfrak{X}_i$ such that $(\mathfrak{X}_i)_\eta \widetilde{\to} X_i$.

Let us say a couple of words on the proof of Fact 5.3.4. The functor η takes morphisms of $\mathcal{B}$ to isomorphisms hence it induces a functor $\mathcal{F} : k^\circ\text{-}Fsch/\mathcal{B} \to$ *st*-*k*-An^c. One easily sees that $\mathcal{F}$ is faithful. The proof that $\mathcal{F}$ is full reduces to proving claim (i) of the above remark. This is essentially a strong version of the Chow lemma and the main ingredient in its proof is Gerritzen-Grauert theorem. Since we know that $\mathcal{F}$ is fully faithful, it remains to show that it is essentially surjective, i.e. to prove claim (ii) of the remark. For a strictly k-affinoid space it is very easy to find a formal model, and in general one finds fixes an affinoid covering $X = \cup_i X_i$, finds models $\mathfrak{X}_i$ and then uses that $\mathcal{F}$ is full to find formal blow ups $\mathfrak{X}'_i \to \mathfrak{X}$ so that the formal schemes $\mathfrak{X}'_i$ glue to a formal model $\mathfrak{X}'$ of X.

5.4. **Rigid geometry.** A naive attempt to construct the generic fiber of an affine formal scheme $\mathfrak{X} = \mathrm{Spf}(A)$ is to declare that $\mathfrak{X}_\eta$ is the set $\mathrm{Spec}(A)\backslash\mathrm{Spf}(A)$ of all non-open ideals. Such definition is not compatible with formal localizations, because the Zariski topology becomes stronger on localizations. In particular, this definition cannot be globalized. The situation improves, however, if one only considers the set of closed points of $\mathrm{Spec}(A)\backslash\mathrm{Spf}(A)$. In some cases such spectrum can be globalized, though the usual Zariski topology should be replaced with a certain G-topology if we want the definition to globalize. We will not develop this point of view, but we will show how a similar approach gives rise to the rigid geometry of Tate.

Let k be a non-archimedean field with a non-trivial valuation. Strictly k-affinoid algebras are the basic objects of rigid geometry over k. An affinoid space $X_0 = \mathrm{Sp}(\mathcal{A})$ is defined as the set of maximal ideals of A provided with the G-topology of finite unions of affinoid domains. By Hilbert Nullstelensatz, the residue field of any point $x \in X_0$ is finite over k and hence X_0 is the set of Zariski closed points of $X = \mathcal{M}(\mathcal{A})$. The theory of rigid affinoid spaces and general rigid analytic spaces is developed similarly to the theory of strictly k-analytic spaces from §§2–3. Some intermediate results are slightly easier to prove because we only worry for Zariski closed points, but in the end one has less tools to solve problems. For example, Shilov boundaries and the class of good spaces are not seen in rigid geometry. Another example of an application where generic points are very important is the theory of étale cohomology of analytic spaces. There exist non-zero étale sheaves which have zero stalks at all rigid points but they necessarily have a non-zero stalk at a point of X. (For coherent sheaves rigid points form a conservative family, and so they are easily tractable in the framework of Tate's rigid geometry.)

5.5. **Adic geometry.**

5.5.1. *Basic definitions.* Adic geometry replaces formal schemes with more general objects that have a honest generic fiber (as an adic space). Let us recall why formal schemes have no generic fiber. Let k be a non-archimedean field with a non-trivial valuation and non-zero $\pi \in k^{\circ\circ}$ and let A be a π-adic k°-algebra of finite type over. Formal inverting of π produces the zero ring in two stages: first we invert π obtaining a strictly k-affinoid algebra $\mathcal{A}$ and then we have to factor over the unit ideal because the π-adic topology on $\mathcal{A}$ is trivial (and so the π-adic separated completion of $\mathcal{A}$ is 0). This suggests to extend the category of adic rings so that the topological rings like $\mathcal{A}$ (with its Banach topology) are included. R. Huber suggested a way to do that, which is very natural if we recall how the topology of k is actually defined.

Definition 5.5.1. An *f-adic ring* is a topological ring that contains an open adic ring A_0 with a finitely generated ideal of definition. Any such A_0 is called a *ring of definition* (because it can be used to define the topology of A).

Note that ring of definition is an analog of a unit ball for a norm.

Exercise 5.5.2. (i) Any k-affinoid algebra $\mathcal{A}$ is f-adic.
(ii) $\mathcal{A}$ is reduced if and only if $\mathcal{A}^\circ$ is a ring of definition.

Adic spectrum is defined analogously to analytic spectrum but using all continuous semivaluations. This forces one to modify the notion of a basic ring as follows.

Definition 5.5.3. (i) An *affinoid ring* is a pair $A = (A^\triangleright, A^+)$ where $A^\triangleright$ is an f-adic ring and A^+ is an open subring which is integrally closed in $A^\triangleright$ and is contained in the ring of power-bounded elements A°. The ring A_+ is called the *ring of integers* of A.
(ii) The adic spectrum $\mathrm{Spa}(A)$ is the set of all equivalence classes of continuous semivaluations on $A^\triangleright$ such that $|a| \le 1$ for any $a \in A^+$.

The (usual!) topology and the structure sheaf on $\mathrm{Spa}(A)$ are defined using rational domains. This is done similarly to our definitions, so we omit the details.

Example/Exercise 5.5.4. (i) The only ring of integers of a strictly k-affinoid algebra $\mathcal{A}$ is $\mathcal{A}^\circ$. The k-affinoid rings in adic geometry are the pairs $(\mathcal{A}, \mathcal{A}^\circ)$.
(ii) For any adic k°-algebra A of finite type, (A, A) is an affinoid ring. Check that $\mathfrak{X} = \mathrm{Spa}(A, A)$ is an adic space which is the disjoint union of the *generic fiber* $\mathfrak{X}_\eta = \mathrm{Spa}(\mathcal{A}, \mathcal{A}^\circ)$ where $\mathcal{A}^\circ = A_\pi$ and the closed fiber $\mathfrak{X}_s$ which consists of all semivaluations that vanish on π. Note that $\mathrm{Spf}(A)$ naturally embeds into $\mathfrak{X}_s$ as the set of points of $\mathfrak{X}$ with trivial valuation on the residue field.
(iii) An *affinoid field* is an affinoid ring k such that $k^\triangleright$ is a valued field of height $h^\triangleright \le 1$ (with the induced topology) and k^+ is a valuation ring of k contained in k°. Let h^+ be the height of k^+. Show that $\mathrm{Spa}(k)$ is a chain (under specialization) of $h^+ + 1 - h^\triangleright$ points.

Remark 5.5.5. (i) Spectra of affinoid fields are "atomic objects" in the sense that they do not admit non-trivial monomorphisms from other spaces. Thus, a point of height at least two or a fiber of a morphism over such point is not an adic space.

(ii) Points of height at least two are very different from the usual analytic points. For example, their local rings are usually not henselian (because there is no reasonable completion for valued fields of height more than one).

Exercise 5.5.6. Describe all adic points of the affine line over k. (Hint: show that the only new points are height two points contained in the closures of type two points $x \in \mathbf{A}_k^{\mathrm{ad}}$. Each connected component of $\mathbf{A}_k^{\mathrm{ad}} \setminus \{x\}$ contains one such point.)

5.5.2. *Comparison of categories and spaces.* In principle, all approaches to non-archimedean analytic geometry produce the same category of spaces of finite type (with rational radii of convergence).

Fact 5.5.7. The following categories are naturally equivalent: (a) the category of compact strictly k-analytic spaces, (b) the category of formal k°-schemes of finite type localized by admissible blow ups, (c) the category of quasi-compact and quasi-separated rigid k-analytic space, (d) the category of quasi-compact and quasi-separated adic $\mathrm{Spa}(k, k^\circ)$-spaces of locally finite type.

Let $\mathfrak{X}$ be a formal k°-scheme of finite type and let $\mathfrak{X}^{\mathrm{rig}}$, $\mathfrak{X}^{\mathrm{an}}$ and $\mathfrak{X}_\eta^{\mathrm{ad}}$ be its generic fibers in the three categories of k-analytic spaces. On the level of topological spaces this objects are related as follows.

Fact 5.5.8. $\mathrm{proj}\lim_{f:\mathfrak{X}'\to\mathfrak{X}} |\mathfrak{X}'| \widetilde{\to} |\mathfrak{X}_\eta^{\mathrm{ad}}| \supset |\mathfrak{X}_\eta^{\mathrm{an}}| \supset |\mathfrak{X}_\eta^{\mathrm{rig}}|$, there the limit is taken over all admissible formal blow ups f. Furthermore, $\mathfrak{X}_\eta^{\mathrm{an}}$ is the set of all points of height one in $\mathfrak{X}_\eta^{\mathrm{ad}}$ and also it is homeomorphic to the maximal Hausdorff quotient of $\mathfrak{X}_\eta^{\mathrm{ad}}$, and $\mathfrak{X}_\eta^{\mathrm{rig}}$ is the set of Zariski closed points of $\mathfrak{X}_\eta^{\mathrm{an}}$.

Finally, the sheaves on these spaces are connected as follows.

Fact 5.5.9. The topoi (i.e. the categories of sheaves of sets) of the following sites are equivalent: $\mathfrak{X}_\eta^{\mathrm{an}}$ with the G-topology of compact strictly k-analytic domains, $\mathfrak{X}_\eta^{\mathrm{rig}}$ with the topology of compact rigid domains, and $\mathfrak{X}_\eta^{\mathrm{ad}}$ with its usual topology. In particular, $\mathfrak{X}^{\mathrm{ad}}$ is simply the set of points of all these sites.

6. Analytic curves

6.1. **Examples.**

Definition 6.1.1. (i) A *k-analytic curve C* is a k-analytic space of pure dimension one, i.e. $\dim(C) = 1$ and C does not contain discrete Zariski closed points.

(ii) In the same way as in Definition 2.3.9 we divide the points of C to four types accordingly to their completed residue field.

6.1.1. *Constructions.* Let us first list some constructions what allow to create/enrich our list of k-analytic curves: (i) analytification of an algebraic curve, (ii) generic fibers of formal curves, (iii) an analytic domain in a curve, (iv) a finite covering of a curve (or, more generally, a covering with discrete fibers).

Example/Exercise 6.1.2. (i) The following curves can be obtained by the first method: affine line, projective curves, affine line with doubled origin.

(ii) Let $\mathcal{X}$ be an irreducible projective algebraic k-curve with $k(\mathcal{X}) = K$. The Zariski closed points of $X = \mathcal{X}^{\mathrm{an}}$ are in one-to-one correspondence with the closed points of $\mathcal{X}$, and other points of X are in one-to-one correspondence with the valuations on K that extend the valuation on k. In particular, $K \hookrightarrow \mathcal{H}(x)$ and $\widehat{K}\widetilde{\to}\mathcal{H}(x)$ for any not Zariski closed point $x \in X$.

(iii) The following curves can be obtained by the second method: compact k-analytic curves.

(iv) Most of Hausdorff curves admit a formal model of locally finite type over k°. For example, find such models for the affine line and for the unit disc.

Now, let us study the other two methods with more details.

6.1.2. *Domains in the affine line.* A typical example of a compact domain X is a closed disc $E(a, r)$ with finitely many removed open discs $E(a_i, r_i)$.

Exercise 6.1.3. (i) Prove that X is a Laurent domain in $E(a, r)$.

(ii) Show that if r and r_1 are linearly independent over $|k^\times|$ then X is not a finite covering of a disc. (Hint: if $X = \mathcal{M}(A)$ is finite over $E(b, s)$ then $\rho(\mathcal{A}) \subset \{0\} \cup \sqrt{s^{\mathbf{Z}}|k^\times|}$.)

(iii) Show that one can extend X a little bit so that $r_i \in \sqrt{r^{\mathbf{Z}}|k^\times|}$ and then X is a finite covering of $E(0, r)$.

Next we describe neighborhoods of points of especially simple form.

Exercise 6.1.4. Assume that $k = k^a$. Show that a point $x \in \mathbf{A}^1_k$ admits a fundamental family of open neighborhoods X_i as follows:

(i) if x is of type 1 or 4 then X_i are open discs,

(ii) if x is of type 3 then X_i are open annuli,

(iii) if x is of type 2 then X_i are open discs with removed finitely many closed discs, and in addition one can achieve that $X_i \setminus \{x\}$ is a disjoint union of open discs and finitely many open annuli.

Finally, let us discuss some open domains in $\mathbf{A}^1_k$ with $k = k^a$.

Exercise 6.1.5. (i) Show that a Hausdorff gluing of two open annuli is an open annulus. (Hint: use Exercise 4.1.13(i).)

(ii) Show that a filtered union of a countable family of annuli does not have to be an annulus. (Hint: take $\mathbf{P}^1_k$ and remove two type 4 points.)

(iii) We say that a non-archimedean field k is *local* if $|k^\times|\widetilde{\to}\mathbf{Z}$ and $\widetilde{k}$ is finite. Show that either k is finite over $\mathbf{Q}_p$ or $k\widetilde{\to}\mathbf{F}_p((t))$. Drinfel'd upper half-plane is defined as $\mathbf{P}^1_k \setminus \mathbf{P}^1_k(k)$. Show that it is an open analytic domain in $\mathbf{P}^1_k$.

6.1.3. *Finite covers.* First, we consider an inseparable cover giving rise to a (mild) pathology that cannot occur in the algebraic world.

Example/Exercise 6.1.6. (i) Construct a non-archimedean field k with $\mathrm{char}(k) = p$ and $[k : k^p] = \infty$.

(ii) Choose elements $a_i \in k$ which are independent over k and such that $|a_i|$ tends to zero. Set $\mathcal{A} = k\{T, S\}/(S^p - \sum_{i=0}^\infty a_i T^{pi}$. Show that $X = \mathcal{M}(\mathcal{A})$ is a finite covering of $E(0, 1)$ of degree p, $X \otimes_k l$ is reduced for any finite field extension l/k but $X\widehat{\otimes}_k k^{1/p}$ is not reduced.

Next, we study some quadratic covers. They will give rise to various interesting examples.

Exercise 6.1.7. Assume that $\mathrm{char}(\widetilde{k}) \neq 2$ and consider a series $f(T) = \sum_{i=0}^\infty a_i T^i \in k\{T\}$ with the affinoid algebra $\mathcal{A} = k\{T, S\}/(S^2 - f(T))$ and the quadratic covering

$\phi : X = \mathcal{M}(\mathcal{A}) \to E(0,1)$. Show that any type 4 point $x \in \mathbf{A}^1_k$ has two preimages, and the maximal point p_r of the disc $E(0,r)$ has two preimages if and only if $|a_0| > |a_i|r^i$ for $i > 0$. In particular, a Zariski closed point x has two preimages if and only if $f(x) \neq 0$. (Hint: show that the binomial expansion of $\sqrt{1+z}$ has radius of convergence 1 over k.)

Now let us study elliptic curves using double covers. For simplicity, we also assume that $k = k^a$.

Example/Exercise 6.1.8. It is known from algebraic geometry that any elliptic curve over k can be realized as the double covering $\phi : E \to \mathbf{P}^1_k$ given by $S^2 = T(T-1)(T-\lambda)$.

(i) Assume that $|\lambda| > 1$.

(a) Show that the points with one preimage are precisely the points of the disjoint intervals $[0,1]$ and $[\lambda, \infty]$. In particular, X contains a cycle $\Delta(E)$ which is the preimage of the interval $I = [p_1, p_{|\lambda|}]$ and a contraction of $\mathbf{P}^1_k$ onto I lifts to the contraction of E onto $\Delta(E)$.

(b) Show that the preimage of the disc $E(0,r)$ with $1 < r < |\lambda|$ is a closed annulus, and deduce that E is glued from two annuli.

(c)* Show that E is glued from $A(0;1,|\lambda|^2$ by identifying $A(0;1,1)$ with $A(0;|\lambda|,|\lambda|)$. Moreover, the universal covering of E is isomorphic to G_m and $G_m/q^{\mathbf{Z}} \widetilde{\to} E$ for an element $q \in k$ with $|q| = |lam|^2$.

(ii) Show that if $|\lambda - 1| < 1$ or $|\lambda| < 1$ then the structure of E is similar but with respect to other intervals connecting the four points.

(iii) Assume that $|\lambda| = |\lambda - 1| = 1$.

(a) Show that the points with one preimage are p_1 and the points of the disjoint intervals $[0, p_1)$, $[1, p_1)$, $[\lambda, p_1)$ and $[\infty, p_1)$. Let $z = \Delta(E)$ be the preimage of p_1 then the contraction of $\mathbf{P}^1_k$ onto p_1 lifts to the contraction of E onto z.

(b) Show that $E \setminus \{z\}$ is a disjoint union of open discs. Furthermore, $\widetilde{\mathcal{H}(z)}$ is of genus one over $\widetilde{k}$ and the closed points of its projective model parameterize the open discs of $E \setminus \{z\}$. In particular, z is not locally embeddable into $\mathbf{A}^1_k$.

The curves from (i) and (ii) are called Tate curves, or elliptic curves with bad reduction. The curves from (iii) are called curves with good reduction. In the sequel by *genus* of a type two point z we mean the algebraic genus of $\widetilde{\mathcal{H}(z)}$ over $\widetilde{k}$. Points of positive genus are very special and very informative.

Exercise 6.1.9. (i) Study curves C of genus two given by $S^2 = f(T)$ with $f(T)$ of degree five. Show that the first Betti number of C plus the sum of genera of its type two points equals to the genus of C.

(ii)* Prove the same for any C given by $S^2 = f(T)$ where $f(T)$ is a polynomial without multiple roots.

Finally, let us construct wild non-compact examples.

Exercise 6.1.10. Assume that k is not discrete valued.

(i) Show that if $|a_i|$ increase and tend to 1 then $f(T) = \sum_{i=0}^{\infty} a_i T^i$ is a function with infinitely many roots on the open unit disc $D(0,1)$.

(ii) Show that by an appropriate choice of $f(T)$ as above one can achieve that the corresponding double cover C of $D(0,1)$ has infinitely many loops and infinitely many positive genus points.

It will follow from some further results that C is an example of a non-compactifiable space. In the sequel we will study compactifiable (mainly, compact) curves.

6.2. **General facts about compact curves.**

6.2.1. *Algebraization.*

Fact 6.2.1. Any proper k-analytic curve X is projective. In particular, X is algebraizable.

Exercise 6.2.2. (i)* Prove the above fact. (Hint: take a Zariski closed point P and show that $H^1(X, \mathcal{O}_X(nP))$ vanishes for large enough n by Kiehl's theorem on direct images 4.2.17. Deduce that a linear system $\mathcal{O}_X(nP)$ with large enough n gives rise to a finite morphism to the projectivization of $H^0(X, \mathcal{O}_X(nP))$. Then use Fact 5.1.6 from GAGA.

(ii) Deduce that the curve from Exercise 6.1.6 is not a domain in a proper curve. Moreover, find k with $[k : k^p] < \infty$ and a finite extension K/K_r such that K is not isomorphic to the completion of a finitely generated k-field of transcendence degree one. Use this to construct a k-affinoid curve that cannot be embedded into a proper curve.

6.2.2. *Compactification.* We saw that if a separated compact curve C is not geometrically reduced then it does not have to be embeddable into a proper one. In the opposite direction we have the following result.

Fact 6.2.3. Any separated geometrically reduced k-analytic curve is isomorphic to a domain in a projective k-curve.

A generic idea of the proof is as follows: we would like to patch the boundary of C, which consists of generic points (of types 2 and 3). One proves that a curve is geometrically reduced at a not Zariski closed point if and only if it is rig-smooth at such point. If C is rig-smooth at x then it admits a quasi-étale morphism $\phi : C \to \mathbf{A}^1_k$ locally around x. If we deform a quasi-étale morphism slightly then the isomorphism class of C does not change. Therefore we can define ϕ using only equations of the form $\sum_{i=0}^n a_i(T)y^i$ where the coefficients a_i are meromorphic. This allows to compactify C at all points of its boundary.

Using the Riemann-Roch theorem on a projective curve one deduces the following corollary.

Fact 6.2.4. A separated, compact, and geometrically reduced k-analytic curve is affinoid if and only if it does not contain proper irreducible components.

6.2.3. *Formal models.* In the sequel we assume that C is a compact geometrically reduced strictly k-analytic rig-smooth curve and the valuation is non-trivial. For any formal model $\mathfrak{C}$ of C let $\mathfrak{C}^0 \subset C$ be the preimage of the set of generic points of $\mathfrak{C}_s$ under the reduction map.

Fact 6.2.5. (i) The set $\mathfrak{C}_0$ determines the formal model $\mathfrak{C}$ up to a finite admissible blow up.

(ii) If C is separated then a finite set V of type 2 points is of the form $\mathfrak{C}_0$ for some formal model if and only if V contains the boundary of C and hits each proper irreducible component of C.

Exercise 6.2.6. (i) Show that (ii) above does not hold in the non-separated case. (Hint: take the closed disc with doubled open disc, and patch in an open annulus instead of the doubled open disc. Then there is no formal model with a single generic point, although such a model exists as a not locally separated formal algebraic space.)

(ii)* Deduce Fact 6.2.5 from Fact 6.2.4.

6.3. **Rig-smooth curves.** In this section For simplicity we assume that k ia algebraically closed. In the general case, all our results hold up to a finite ground field extension.

6.3.1. *Geometric structure of analytic curves.* Here is the main result about the structure of C in its geometric formulation. An equivalent approach via formal models will be discussed later.

Fact 6.3.1. There exist a finite set V of type two points such that $C \setminus V$ is a disjoint union of open discs and finitely many open annuli.

This claim is very strong and implies many other important results that we state as exercises.

Exercise 6.3.2. (i) C has finitely many points of positive genus and C can be contracted onto its subset $\Delta(V)$ homeomorphic to a finite graph. (Hint: take $\Delta(V)$ to be the the union of V and the open intervals through the annuli then $C \setminus \Delta(V)$ is a disjoint union of open unit discs which can be easily contracted.)

(ii)* If C is proper then its algebraic genus equals to the sum of the genera of type two points plus the first Betti number of $\Delta(V)$. (Hint: use the semistable formal model associated to V in the next section.)

First let us study what is the freedom in the choice of V.

Exercise 6.3.3. (i) Cofinality: show that any finite set of type two points can be enlarged to a set V as above.

(ii) Conjecture 4.3.1 holds true for curves. (Hint: the sets $\Delta(V)$ form the required filtered family.)

(iii)* Minimality: show that there exists a minimal such V unless $\mathbf{P}^1_k$ or a Tate curve is a connected component of C. (Hint: use exercise 4.1.13. Note that the degenerate cases are proper curves that can be covered by annuli.)

Next, let us describe the local structure of C.

Exercise 6.3.4. Show that a point $x \in C$ has a fundamental family of open neighborhoods X_i such that

(i) X_i are open discs when x is of type 1 or 4,

(ii) X_i are open annuli when x is of type 2,

(iii) $X_i \setminus \{x\}$ are disjoint unions of open discs and finitely many open annuli when x is of type 2.

Using gluing of annuli and discs from Fact 4.1.13, it is easy to show that the above local description of C is equivalent to its global description. This local fact, in its turn, easily reduces to study of the field $\mathcal{H}(x)$. For example, for types 3 and 4 it suffices to show that $\mathcal{H}(x) = \widehat{k(T)}$ is topologically generated by an element. Surprisingly, no simple proof of this fact is known. A shortest currently known proof can be found in [Tem4]. As a result one obtains a new proof of the stable reduction theorem (which, as we will see, is essentially equivalent to Fact 6.3.1.

6.3.2. *Semistable formal models.*

Definition 6.3.5. A formal k°-scheme is *semistable* if it is étale-locally isomorphic to the formal schemes of the form $\mathfrak{Z}_{n,a} = \mathrm{Spf}(k^\circ\{T_1, \dots, T_n\}/(T_1, \dots, T_n - a))$ with $a \in k^\circ$. A *polystable* formal k°-scheme is a formal scheme that is étale-locally isomorphic to a product of semistable formal k°-schemes.

Let $\mathfrak{X}$ be normal in its generic fiber. A general theory of *formal fibers* (i.e. the fibers of reduction maps) predicts that $\mathfrak{X}$ is polystable if and only if it has the same formal fibers as products of the model schemes $\mathfrak{Z}_{n,a}$, see Fact 5.2.17. Applying this to curves one obtains the following result (which can also be proved by an elementary computation).

Exercise 6.3.6. A formal k°-curve $\mathfrak{C}$ with rig-smooth fiber is semistable if and only if the formal fibers over its closed points are open discs (over the smooth points) and open annuli (over the double points of $\mathfrak{C}_s$).

This exercise and Fact 6.2.5 imply that the global description of C given by Fact 6.3.1 is equivalent to the following fundamental result, which can be proved by a classical (but rather complicated) algebraic theory that involves stable reduction over a discrete valued field and the theory of moduli spaces of curves.

Fact 6.3.7 (Semistable reduction theorem for analytic curves)**.** Any compact rig-smooth strictly analytic curve over an algebraically closed field k admits a semistable formal model.

In the same way, Exercise 6.3.3 implies the following generalization of the above fact.

Exercise 6.3.8. (i) Cofinality: any formal model of C admits an admissible blow up that is semistable.

(ii) Stable reduction theorem: if $\mathbf{P}^1_k$ and a Tate curve are not components of C then it possesses a minimal semistable formal model (called formal stable model).

6.4. **Skeletons.** The reader that solved Exercises in §6.3.1 is probably familiar with part of the ideas of this section. All facts of this section follow easily from the results of §6.3.1.

Definition 6.4.1. (i) Let V_0 be a finite set of type 1 and 2 points of C. The *skeleton* $\Delta(C, V_0)$ is defined as follows: its set of vertexes V is the set of points $x \in C$ that are not contained in an open annulus $A \subset C \setminus V_0$, and its edges are formed by the points $x \in C$ that are not contained in an open disc disjoint from V_0. A vertex is *infinite* if it is of type 1.

(ii) The skeleton $\Delta = \Delta(C, V_0)$ is *degenerate* if the set V_f of finite vertexes is empty.

For the sake of completeness, we make a remark about the more general situation that we will not study.

Remark 6.4.2. The definition makes sense for any V and any curve over $k = k^a$. For a general curve the skeleton is defined as the image of a skeleton after an appropriate ground field extension.

Exercise 6.4.3. Keep our assumptions on C and k and assume that C is connected.

(i) Show that Δ is a finite graph whose infinite vertexes are the points of V_0 of type 1.

(ii) Show that the only degenerate cases are when C is a Tate curve and V_0 is empty, or $C \widetilde{\to} \mathbf{P}^1_k$ and V_0 is a set of at most two type 1 points. Thus, in the degenerate cases Δ is empty, an infinite vertex, an interval with infinite ends, or a loop without vertexes.

(iii) Show that in the non-degenerate case V_f is the minimal set of points such that $C \setminus V_f$ is a disjoint union of open discs and annuli such that annuli are disjoint from V_0 and an open disc contains at most one infinite point of V_0.

(iv)* Show that if C is not proper or V_0 has finite vertexes then the above description of Δ implies (and is equivalent to) the following stable modification theorem: if $\mathfrak{C}$ is a formal rig-smooth k°-curve with a generically reduced Cartier divisor $\mathfrak{D}$ then there exists a minimal *modification* $\mathfrak{C}' \to \mathfrak{C}$ (i.e. a proper morphism whose generic fiber is an isomorphism) such that $\mathfrak{C}'$ is semistable and the strict transform of $\mathfrak{D}$ is étale over $\mathrm{Spf}(k^\circ)$.

(v)* Formulate and prove the analogous statement when C is proper and V_0 has no finite vertexes. (This is the stable reduction theorem for a formal curve with divisor.)

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Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Giv'at Ram, Jerusalem, 91904, Israel
E-mail address: temkin@math.huji.ac.il